Tropical arithmetic and Graph algorithms

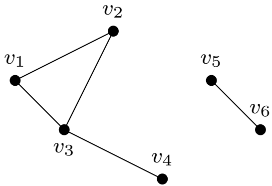
Based in part on lectures by Lorenzo Ciardo (edited by Kristin Shaw)

MAT2250 - University of Oslo

The distance matrix of a graph

Definition

Given two vertices v_i, v_j in a graph G, their **distance** $d(v_i, v_j)$ is the length of a shortest path connecting v_i to v_j and is equal to ∞ if no such path exists.
The **distance matrix** $\Delta(G)$ is the $n \times n$ matrix whose (i, j)'th entry is $d(v_i, v_j)$.

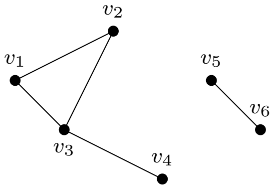


$$\Delta(G) = \begin{bmatrix} 0 & 1 & 1 & 2 & \infty & \infty \\ 1 & 0 & 1 & 2 & \infty & \infty \\ 1 & 1 & 0 & 1 & \infty & \infty \\ 2 & 2 & 1 & 0 & \infty & \infty \\ \infty & \infty & \infty & \infty & 0 & 1 \\ \infty & \infty & \infty & \infty & 1 & 0 \end{bmatrix}$$

The distance matrix of a graph

Definition

Given two vertices v_i, v_j in a graph G, their **distance** $d(v_i, v_j)$ is the length of a shortest path connecting v_i to v_j and is equal to ∞ if no such path exists.
The **distance matrix** $\Delta(G)$ is the $n \times n$ matrix whose (i, j)'th entry is $d(v_i, v_j)$.



$$\Delta(G) = \begin{bmatrix} 0 & 1 & 1 & 2 & \infty & \infty \\ 1 & 0 & 1 & 2 & \infty & \infty \\ 1 & 1 & 0 & 1 & \infty & \infty \\ 2 & 2 & 1 & 0 & \infty & \infty \\ \infty & \infty & \infty & \infty & 0 & 1 \\ \infty & \infty & \infty & \infty & 1 & 0 \end{bmatrix}$$

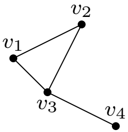
Facts

- $\Delta(G)$ is symmetric.
- $\Delta(G)$ has zeros on its main diagonal.

Is there a relation between the adjacency and the distance matrix of a graph?

Powers of the adjacency matrix

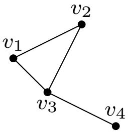
Is there a relation between the adjacency and the distance matrix of a graph?



$$A_G = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \Delta(G) = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{bmatrix}$$

Powers of the adjacency matrix

Is there a relation between the adjacency and the distance matrix of a graph?



$$A_G = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \Delta(G) = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{bmatrix}$$

Facts

Recall: the k-th power of A_G gives the number of paths between v_i and v_j of length k.

$$A_G^2 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \qquad A_G^3 = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \\ 4 & 4 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

Tropical algebra

Consider the set $\bar{\mathbb{R}} = \mathbb{R} \cup \infty$ with the operations

$$x \oplus y = \min(x, y) \qquad \text{and} \qquad x \otimes y = x + y.$$

Tropical algebra

Consider the set $\bar{\mathbb{R}} = \mathbb{R} \cup \infty$ with the operations

$$x \oplus y = \min(x, y) \qquad \text{and} \qquad x \otimes y = x + y.$$

Notice ∞ is the "tropical zero" (unit for $\oplus$) and 0 is the "tropical one" (unit for $\otimes$).
$(\bar{\mathbb{R}}, \oplus, \otimes)$ is an *semi-field* - elements do not have additive inverses, yet all other field axioms are satisfied.

Tropical matrix multiplication

Given matrices $A, B \in \bar{\mathbb{R}}^{n,n}$, we define the sum $A \oplus B \in \bar{\mathbb{R}}^{n,n}$ and product $A \otimes B \in \bar{\mathbb{R}}^{n,n}$:

- $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} = \min(a_{ij}, b_{ij})$;
- $(A \otimes B)_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes b_{kj} = \min_{k=1,\ldots,n}(a_{ik} + b_{kj})$.

Observation

$$\mathcal{Z} = \begin{bmatrix} \infty & \infty & \dots & \infty \\ \infty & \infty & \dots & \infty \\ \vdots & \vdots & \ddots & \vdots \\ \infty & \infty & \dots & \infty \end{bmatrix}$$

tropical zero matrix.

$$\mathcal{E} = \begin{bmatrix} 0 & \infty & \dots & \infty \\ \infty & 0 & \dots & \infty \\ \vdots & \vdots & \ddots & \vdots \\ \infty & \infty & \dots & 0 \end{bmatrix}$$

tropical identity matrix.

Observation

$$\mathcal{Z} = \begin{bmatrix} \infty & \infty & \dots & \infty \\ \infty & \infty & \dots & \infty \\ \vdots & \vdots & \ddots & \vdots \\ \infty & \infty & \dots & \infty \end{bmatrix}$$

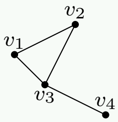
tropical zero matrix.

$$\mathcal{E} = \begin{bmatrix} 0 & \infty & \dots & \infty \\ \infty & 0 & \dots & \infty \\ \vdots & \vdots & \ddots & \vdots \\ \infty & \infty & \dots & 0 \end{bmatrix}$$

tropical identity matrix.

Definition

The **tropical adjacency matrix** $\mathcal{A}(G)$ of a graph G is obtained from $A(G)$ by substituting 0 with ∞.



$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathcal{A}(G) = \begin{bmatrix} \infty & 1 & 1 & \infty \\ 1 & \infty & 1 & \infty \\ 1 & 1 & \infty & 1 \\ \infty & \infty & 1 & \infty \end{bmatrix}$$

The distance matrix of a graph

Theorem

The distance matrix of a graph $G = (V, E)$ with $|V| = n$ is given by

$$\Delta(G) = (\mathcal{E} \oplus \mathcal{A}(G))^{\otimes (n-1)}.$$

The distance matrix of a graph

Theorem

The distance matrix of a graph $G = (V, E)$ with $|V| = n$ is given by

$$\Delta(G) = (\mathcal{E} \oplus \mathcal{A}(G))^{\otimes(n-1)}.$$

$$\mathcal{A}(G) = \begin{bmatrix} \infty & 1 & 1 & \infty \\ 1 & \infty & 1 & \infty \\ 1 & 1 & \infty & 1 \\ \infty & \infty & 1 & \infty \end{bmatrix} \qquad \mathcal{E} \oplus \mathcal{A}(G) = \begin{bmatrix} 0 & 1 & 1 & \infty \\ 1 & 0 & 1 & \infty \\ 1 & 1 & 0 & 1 \\ \infty & \infty & 1 & 0 \end{bmatrix}$$

The distance matrix of a graph: proof

Theorem

The distance matrix of a graph $G = (V, E)$ with $|V| = n$ is given by

$$\Delta(G) = (\mathcal{E} \oplus \mathcal{A}(G))^{\otimes(n-1)}.$$

The distance matrix of a graph: proof

Theorem

The distance matrix of a graph $G = (V, E)$ with $|V| = n$ is given by

$$\Delta(G) = (\mathcal{E} \oplus \mathcal{A}(G))^{\otimes(n-1)}.$$

Proof.

- Call $\mathcal{A} = \mathcal{A}(G)$.

$$(\mathcal{E} \oplus \mathcal{A})^{\otimes(n-1)} = \bigoplus_{0 \leq r \leq n-1} \binom{n-1}{r} \mathcal{A}^{\otimes r} \otimes \mathcal{E}^{\otimes(n-1-r)} = \bigoplus_{0 \leq r \leq n-1} \binom{n-1}{r} \mathcal{A}^{\otimes r}$$

$$= \mathcal{E} \oplus \mathcal{A} \oplus \mathcal{A}^2 \oplus \cdots \oplus \mathcal{A}^{n-1}.$$

The distance matrix of a graph: proof

Theorem

The distance matrix of a graph $G = (V, E)$ with $|V| = n$ is given by

$$\Delta(G) = (\mathcal{E} \oplus \mathcal{A}(G))^{\otimes(n-1)}.$$

Proof.

- Call $\mathcal{A} = \mathcal{A}(G)$.

$$(\mathcal{E} \oplus \mathcal{A})^{\otimes(n-1)} = \bigoplus_{0\le r\le n-1} \binom{n-1}{r} \mathcal{A}^{\otimes r} \otimes \mathcal{E}^{\otimes(n-1-r)} = \bigoplus_{0\le r\le n-1} \binom{n-1}{r} \mathcal{A}^{\otimes r}$$
$$= \mathcal{E} \oplus \mathcal{A} \oplus \mathcal{A}^2 \oplus \cdots \oplus \mathcal{A}^{n-1}.$$

- Claim: $\left(\mathcal{A}^{\otimes r}\right)_{ij} = r$ if $\exists$ a length r path from i to j, otherwise $\left(\mathcal{A}^{\otimes r}\right)_{ij} = \infty$.

The distance matrix of a graph: proof

Theorem

The distance matrix of a graph $G = (V, E)$ with $|V| = n$ is given by

$$\Delta(G) = (\mathcal{E} \oplus \mathcal{A}(G))^{\otimes(n-1)}.$$

Proof.

- Call $\mathcal{A} = \mathcal{A}(G)$.

$$(\mathcal{E} \oplus \mathcal{A})^{\otimes(n-1)} = \bigoplus_{0 \le r \le n-1} \binom{n-1}{r} \mathcal{A}^{\otimes r} \otimes \mathcal{E}^{\otimes(n-1-r)} = \bigoplus_{0 \le r \le n-1} \binom{n-1}{r} \mathcal{A}^{\otimes r}$$
$$= \mathcal{E} \oplus \mathcal{A} \oplus \mathcal{A}^2 \oplus \cdots \oplus \mathcal{A}^{n-1}.$$

- Claim: $\left(\mathcal{A}^{\otimes r}\right)_{ij} = r$ if $\exists$ a length r path from i to j, otherwise $\left(\mathcal{A}^{\otimes r}\right)_{ij} = \infty$.
  True for $r = 0, 1$. Suppose true for r.

$$\left(\mathcal{A}^{\otimes(r+1)}\right)_{ij} = (\mathcal{A}^{\otimes r} \otimes \mathcal{A})_{ij} = \bigoplus_{k=1}^{n} \left((\mathcal{A}^{\otimes r})_{ik} \otimes \mathcal{A}_{kj} \right) = \min_{k=1,\ldots,n} \left((\mathcal{A}^{\otimes r})_{ik} + \mathcal{A}_{kj} \right).$$

The distance matrix of a graph: proof

Theorem

The distance matrix of a graph $G = (V, E)$ with $|V| = n$ is given by

$$\Delta(G) = (\mathcal{E} \oplus \mathcal{A}(G))^{\otimes(n-1)}.$$

Proof.

- Call $\mathcal{A} = \mathcal{A}(G)$.

$$(\mathcal{E} \oplus \mathcal{A})^{\otimes(n-1)} = \bigoplus_{0 \le r \le n-1} \binom{n-1}{r} \mathcal{A}^{\otimes r} \otimes \mathcal{E}^{\otimes(n-1-r)} = \bigoplus_{0 \le r \le n-1} \binom{n-1}{r} \mathcal{A}^{\otimes r}$$
$$= \mathcal{E} \oplus \mathcal{A} \oplus \mathcal{A}^2 \oplus \cdots \oplus \mathcal{A}^{n-1}.$$

- Claim: $\left(\mathcal{A}^{\otimes r}\right)_{ij} = r$ if $\exists$ a length r path from i to j, otherwise $\left(\mathcal{A}^{\otimes r}\right)_{ij} = \infty$. True for $r = 0, 1$. Suppose true for r.

$$\left(\mathcal{A}^{\otimes(r+1)}\right)_{ij} = (\mathcal{A}^{\otimes r} \otimes \mathcal{A})_{ij} = \bigoplus_{k=1}^{n} \left((\mathcal{A}^{\otimes r})_{ik} \otimes \mathcal{A}_{kj}\right) = \min_{k=1,\ldots,n} \left((\mathcal{A}^{\otimes r})_{ik} + \mathcal{A}_{kj}\right).$$

The distance matrix of a graph: proof

Let $w\colon E \to \mathbb{R}^+$ be a weight function on the edges of G.

Theorem

The weighted distance matrix of a graph $G = (V, E)$ with $|V| = n$ is given by

$$\Delta_w(G) = (\mathcal{E} \oplus \mathcal{A}_w(G))^{\otimes(n-1)}.$$

Proof cont.

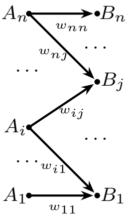
Then the i, j-th entry of the matrix $\mathcal{E} \oplus \mathcal{A} \oplus \mathcal{A}^2 \oplus \cdots \oplus \mathcal{A}^{n-1}$. is the minimum over the lengths of all paths from v_i to v_j. I.e. the distance from v_i to v_j. ■

Computing the lengths of the shortest paths in this way is known as the Floyd–Warshall algorithm. It is an $O(|V|^3)$ algorithm.

Recall that we saw Dijkstra's algorithm which for each vertex returns a spanning tree containing the shortest path to any other vertex. Dijkstra's is an $O(|V|^2)$ algorithm.

Optimal matchings and tropical determinants

Recall the previously considered optimal matching problem for bipartite graphs:
Let $G = K_{n,n}$ and let $w\colon E \to \mathbb{R}^+$ be a weight function. We want to find a matching M with $|M| = n$ of minimal weight.



The **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

The **tropical determinant** of an $n \times n$ matrix $\mathcal{A} = [w_{ij}] \in \overline{\mathbb{R}}^{n \times n}$ is

$$\det_{trop}(\mathcal{A}) := \bigoplus_{\sigma \in S_n} w_{1\sigma(1)} \otimes w_{2\sigma(2)} \otimes \cdots \otimes w_{n\sigma(n)}. \qquad (*)$$

Optimal matchings and tropical determinants

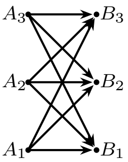
Recall the previously considered optimal matching problem for bipartite graphs:
Let $G = K_{n,n}$ and let $w\colon E \to \mathbb{R}^+$ be a weight function. We want to find a matching M with $|M| = n$ of minimal weight.

The **tropical determinant** of an $n \times n$ matrix $\mathcal{W} = [w_{ij}] \in \overline{\mathbb{R}}^{n\times n}$ is

$$\det_{trop}(\mathcal{W}) := \bigoplus_{\sigma \in S_n} w_{1\sigma(1)} \otimes w_{2\sigma(2)} \otimes \cdots \otimes w_{n\sigma(n)}. \quad (*)$$

The weight of an optimal matching is the tropical determinant of the matrix of weights!
Moreover, an optimal matching corresponds to a permutation σ for which

$$\det_{trop}(\mathcal{W}) = w_{1\sigma(1)} \otimes w_{2\sigma(2)} \otimes \cdots \otimes w_{n\sigma(n)}.$$



Computing the tropical determinant directly requires $O(|V|^4)$ steps.
Carrying out the Hungarian method requires $O(|V|^3)$ steps.

A bit of tropical geometry

For $n = 3$ we can find an optimal matching by drawing a picture of “tropical lines”.

A bit of tropical geometry

For $n = 3$ we can find an optimal matching by drawing a picture of "tropical lines".

$$\mathcal{W} = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \rightarrow \quad \mathcal{W}' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & -2 & -1 \end{pmatrix}$$

Exercises

1. Express the row and column manipulations of the Hungarian algorithm as a tropical version of familiar row and column operations. Do the other usual matrix row and column operations translate to operations on tropical matrices that do not effect $\det_{trop}(\mathcal{A})$?
2. Show that a matrix $\mathcal{A} \in \overline{\mathbb{R}}^{n \times n}$ has tropical determinant $\neq \infty$ if and only if each row and each column contains at least one finite coefficient.
3. A matrix $\mathcal{A} \in \overline{\mathbb{R}}^{n \times n}$ is said to be tropically singular if either $\det_{trop}(\mathcal{A}) = \infty$ or there are at least two permutations $\sigma_1 \neq \sigma_2$ such that

   $$\det_{trop}(\mathcal{A}) = w_{1\sigma_i(1)} \otimes w_{2\sigma_i(2)} \otimes \cdots \otimes w_{n\sigma_i(n)}.$$

   Give an example of a tropically singular matrix with $\det_{trop}(\mathcal{A}) \neq \infty$.
4. For $n = 3$ what does the tropical line arrangement of a singular matrix look like? Can you still determine from the line arrangement the collections of permutations which compute the tropical determinant?