MAT2250 Exercise list
Spring 2021

2. Week 2

Problems from Aiger: 1.8, 1.12, 1.15, 1.26, 1.32, 1.37, 1.39

3. Week 3

Problems from Aigner: 1.9, 1.11,1.13, 1.14

Hint for 1.11: A graphical approach is to draw a partition of n with k parts as Young diagrams (see screencasts or look it up). Group together the integer partitions based on how many parts remain when you remove the first column of the boxes. Now count the number of partitions of each of these types.
Hint for 1.14: The Fibonnaci numbers (probably known to Pingala $\approx$ 2000 years earlier) are defined by $F_n = F_{n-1} + F_{n-2}$ and $F_0 = 0$ $F_1 = 1$. Check the first cases and proceed by induction.

Problem 3.1.
1) Construct a map from the set of permutations of $\{1, \ldots, n\}$ with k cycles to the set of set partitions of $\{1, \ldots, n\}$ with k parts.
2) Construct a map from the set of set partitions of $\{1, \ldots, n\}$ with k parts to the set of integer partitions of n with k parts.
Hint: To come up with both maps be forgetful!
3) Show that both maps are surjective and conclude that $P_{n,k} \leq S_{n,k} \leq \sigma_{n,k}$ for all n, k.
4) Describe the composition of the two maps which sends the set of permutations of $\{1, \ldots, n\}$ with k cycles to integer partitions of n with k parts. What does this have to do with the cycle type described in Aigner?

Problem 3.2.
1) How many ways are there of putting 15 students in a discussion session into 3 (unordered) break out rooms? Describe your answer as a counting coefficient we have seen (do NOT give a numerical answer!).
2) How many ways are there of putting the 15 students into (unordered) break out rooms if the rooms are to contain 6, 5, and 4 students? What if they were to contain 5, 5, 5 students?
3) Generalise your formula to show that the number of ways of partitioning the set $\{1, \ldots, n\}$ into k sets where you have b_i sets of size i is:

$$\frac{n!}{b_1!b_2!\ldots b_n!(1!)^{b_1}\ldots(n!)^{b_n}}$$

Hint: Adapt the argument on page 16 of Aigner.
4) Write a formula for $S_{n,k}$ similar to the one for $\sigma_{n,k}$ on page 16 of Aigner.

4. Week 4

Problems from Aigner: 1.51, 2.8, 2.9, 2.38

Problem 4.1. *Show that if you choose a binary word of length n at random with uniform probability then the expected number of 0's in the word is $\frac{n}{2}$.*
Hint: Use the derivative of $(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.

Problem 4.2. *Let $\pi = \pi_1 \dots \pi_n$ be a permutation of $\{1, \dots, n\}$ written as a word. The index i is called an excedance of π if $\pi_i > i$. What is the expected number of excedances of a permutation of $\{1, \dots, n\}$ selected with the uniform distribution?*
Hint: Use that if X is the number of excedances of a permutation, then $X = X_1 + \dots + X_n$ where $X_i(\pi) = 1$ if i is an excedance of π and $X_i(\pi) = 0$ otherwise.

Problem 4.3. *Consider selecting partitions of n with uniform probability. Let $X : Part_n \to \mathbb{R}$ be the size of the first part and $Y : Part_n \to \mathbb{R}$ be the number of parts. Find $E(X - Y)$*
Hint: The size of the first part of a partition is equal to the number of parts of its reflection. *Use this to organize the summands in the average.*

5. Week 5

Problems from Aigner: 2.11, 2.13*, 2.42, 2.43, 3.9, 3.15, 3.18 (do 5.1 first), 3.36 (do 5.2 first)

Problem 5.1. *Use inclusion-exclusion to prove that*

$$|Surj(N, K)| = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n,$$

where $|N| = n$ and $|K| = k$.

Problem 5.2 (Partial sums of series)**.** *Let $A(z) = \sum_{n \geq 0} a_n z^n$ and*

$$B(z) = \sum_{n \geq 0} b_n z^n = \frac{A(z)}{1 - z}.$$

What is the sequence $b_0, b_1, b_2, \dots$?
Use this and the series expansion of $\log(1+z)$ to solve exercise 3.18 of Aigner.

Problem 5.3 (enumerating words)**.**

1. *Let a_n be the number of words of length n on an alphabet of size k (set $a_0 = 1$). Express the generating function $F_k(z) = \sum_{n \geq 0} a_n$ as a rational function. Eg, when $k = 2$, we are talking about binary words, and $a_n = 2^n$.*
2. *Let $E(z)$ be the generating function for the number of words of length n with an even number of zeros and $O(z)$ be the number of binary words of length n with an odd number of zeros. Show that $O(z) = F_2(z) - E(z)$.*
3. *Solve 3.36 a) (hint: set up a recurrence by considering the two cases for the first digit)*
4. *Solve 3.36 b) (hint: set up a recurrence by considering the three cases for the first digit)*

6. Week 6

Problems from Aigner: 3.10, 3.11

Problem 6.1.
Let $P_n^{1,2}$ be the number of partitions of n using only parts of size 1 and 2 with $P_0^{1,2} := 1$.

(1) *Calculate by hand $P_n^{1,2}$ for $n \le 5$.*
(2) *Find the generating function for the sequence $P_n^{1,2}$.*
(3) *Give an explicit formula for $P_n^{1,2}$. Hint: Use partial fraction decompositions to express the generating function from above in the form*

$$\frac{A}{1-z} + \frac{B}{(1-z)^2} + \frac{C}{1+z}$$

.

Problem 6.2. *A musical consists of n scenes and is to be split into 2 acts by an intermission with refreshments. There is to be one song performed during some scene during the first act and two songs to be performed in two different scenes in the second act. Let f_n be the number of ways of organising the production with n scenes. (It's perfectly fine if the refreshments are served before or after the production, meaning there are no scenes in the first act or no scenes in the second act, respectively.).*

(1) *Show that $f_n = \sum_{k=0}^{n} k\binom{n-k}{2}$. Hint: Think of k as the number of scenes in the first act.*
(2) *Write the generating function of f_n as a product of other generating functions (page 66 of Aigner will help here again).*

Problem 6.3. *Let T_n be the number of ways of triangulating a labeled $(n+2)$-gon (see end of the Catalan numbers screencast). Set $T_0 := 1$. Explain a recurrence formula for T_n and conclude that $T_n = C_n$ where C_n is the n-th Catalan number. Hint: In a triangulation consider the triangle that has vertices, 1, 2, k (where k is determined by the triangulation) and think about "removing" this triangle like we did with the bracket expressions.*

7. Week 7

Problems from Aigner: 5.1, 5.7, 5.16, 5.22, 5.31
In 5.22 by "Estimating the rate of growth", it means find a function to which it is *asymptotically equal* to (by using Stirling's formula).

Problem 7.1. *Use your solution 5.22 to then estimate the rate of growth of the Catalan numbers.*

8. Week 8

Problems from Aigner: 6.1, 6.3, 6.12, 6.14, 6.27
See page 127 for the definition of a cut edge.

9. Week 9

Problems from Aigner: 6.35, 6.43, 7.4, 7.5, 7.15

Problem 9.1. *Prove that the number of acyclic orientations of the complete graph K_n is $n!$ by constructing a bijection between the acyclic orientations and permutations of $\{1, \ldots, n\}$. Hint: Represent a permutation by a word and think of this word as inducing a new (and strange) ordering on the set $\{1, \ldots, n\}$. For example, the identity permutation gives the usual ordering $1 < 2 < 3 < \cdots < n$. But the permutation 23451 gives the* *ordering $2 < 3 < 4 < 5 < 1$. Given an ordering of the vertices Aigner describes how to construct an acyclic orientation of the graph on page 131. Use this to get a map from permutations to acyclic orientations. What is it's inverse?*

10. Week 10

7.23, 7.24, 7.25, 7.29*, 7.30
For 7.29, if you prefer linear algebra over the real numbers to linear algebra over the field $\{0, 1\}$, you can use the directed incidence matrix of any orientation of the graph G.

11. Week 11

8.1, 8.30
(Only two problems for Week 11 because of the obligatory assignment!)

12. Week 14

8.31, 8.13, 8.14, 8.42, 8.43

13. Week 15

8.20, 8.44, 8.45, 8.46

14. Week 16

12.3, 12.5, 12.20, 12.30, 12.31
Optional problems: 12.21, 12.22 (another way of phrasing 12.22, is to say that -1 is a square in $\mathbb{Z}_p$ for p prime when $p \equiv 1 \mod 4$ and it -1 is not a square in $\mathbb{Z}_p$ when $p \equiv 3 \mod 4$).

Problem 14.1. *Show that for a prime p such that $p \equiv 3 \mod 4$ then the elements in $\mathbb{Z}_p \backslash 0$ which are squares in $\mathbb{Z}_p$ are exactly those which are fourth powers in $\mathbb{Z}_p$. In other words,*

$$\{a \in \mathbb{Z}_p \mid a = b^2 \in \mathbb{Z}_p\} = \{a \in \mathbb{Z}_p \mid a = b^4 \in \mathbb{Z}_p\}$$

15. Week 17

12.41 (done in screencast), 12.44, 12.45, 12.49

Problem 15.1. *Show that there exists a finite projective plane of order n if and only if there exist $n-1$ orthogonal latin squares of order n.*
(Hint: Consider two points x, y of the plane and the line L passing through them. Enumerate the other lines through x by $L_1, \dots L_n$ and the lines through y by $J_1, \dots, J_n$ (draw a cartoon picture). Think about the n^2 points of the plane that are not contained on L as the n^2 entries of a latin square where the point p_{ij} corresponds to entry (i, j) if p_{ij} is on lines L_i and J_j. The L line contains $n-1$ other points and there are exactly n lines through each of these points. Use each of these $n-1$ points to come up with a latin square so the $n-1$ squares are mutually orthogonal.)

16. Week 18

13.6, 13.8, 13.9, 13.12, 13.31

17. Week 19

13.16, 13.17, 13.46 (what is the generating polynomial of the dual code?)
Using the binary cyclic code of length 7 generated by $g(x) = x^3 + x + 1$ encode the message $(1, 0, 1, 1)$ following page 296. Using the same binary code you receive the transmission $(0, 0, 1, 1, 1, 0, 1)$. Has an error occurred? If not, what is the decoded the message?

18. Week 20

14.9, 14.10, 14.22, 14.23, 14.27
The importance of Exercise 14.23 is to understand ElGamal encryption.