ENUMERATIVE COMBINATORICS Q & A

(1) **How do we see which method one should use to find out something about $f(n)$?**

This is a great question and a very difficult one to answer, but I will give it a shot. The point is that there is not always a straight path (aka algorithm) to determining a sequence or even figuring something out about it. As a general rule, always try to write down the first few terms. This is good to do even if you end up finding a formula for $f(n)$. In which case you should always go back and "sanity check" (compare your formula to your first calculations).

We've seen many techniques to investigate sequences and often times we will need more than one of these when analysing just one sequence:

(a) Summation Formula
(b) Product Formula
(c) Bijection
(d) Induction
(e) Recurrence
(f) Inversion
(g) Inclusion-Exclusion
(h) Generating Functions and Partial Fraction Decomposition
(i) Exponential Generating Functions
(j) Asymptotic analysis

Often the methods to try first depend on how the sequence is delivered to us. Here are some guidelines to help (they are guidelines not rules!)

Is it an enumerative problem? For instance, the number of binary words satisfying conditions, or the number of subsets satisfying conditions. Then most often we have used a combination of summation and bijection to determine a recurrence relation. Sometimes obtaining a recurrence is as far as we can get, like with the Stirling numbers for instance. But if you have a recurrence relation you can try to proceed to the next step. Inclusion-Exclusion was also a tool we used to answer enumerative problems and actually we can think of this as a stronger version of the summation formula. The summation formula tells us how to count a set when it can be partitioned into disjoint subsets. If you can write a set as a union of subsets which are not disjoint, you can try to determine the size of the set using inclusion-exclusion. This requires controlling the sizes of the subsets and *all* of their intersections.

Is the sequence given as a recurrence? If you can make a decent guess at the formula then you can try to go ahead and prove it using induction. It isn't always possible to guess though, take for example the closed formula for the Fibonacci numbers (good luck guessing the $\sqrt{5}$'s and the golden ratios). Depending on the recurrence formula we could sometimes go further using generating functions and partial fractions and get a closed formula. For example, you can check if your sequence fits the template in Theorem 3.1.

If you have non-constant coefficients in the recurrence, sometimes you can still get lucky and the generating function might be a rational function (a quotient of two polynomials). If the recurrence has non-constant coefficients (depending on n) and in solving for the generating function you get stuck, it's worth giving the exponential generating function a try. Sometimes even with a generating function we can't get an explicit formula, take for instance the number of integer partitions of n.

Can you find a formula for another known sequence $g(n)$ in terms of $f(n)$? Here is a good place to try out the black magic of inversion!

Sometimes it's difficult to say much of anything precise about a sequence. In this situation, you might still be able to figure out something about the asymptotic behaviour. For example, Theorem 5.2 gives us asymptotic estimates for recurrences of the form $T(n) = aT(n/b) + f(n)$ and $T(1) = c$, without giving us a means of solving these recurrences for any choice of a, b, c, and f_n. We saw how to solve some examples, but we do not have a general method. This theorem though we didn't see a prove gives us a way to analyse the asymptotics of the sequence $T(n)$ in terms of the asymptotics of $f(n)$.

It's good to keep in mind that often more than one method works! Remember that we counted derangements using Inversion, Inclusion Exclusion, and Exponential Generating Functions.

(2) **Aigner's exercise 1.5 (on p. 33; not listed in the weekly exercises):**

In the parliament of country X there are 151 seats and three political parties. How many ways (i,j,k) are there of dividing up the seats such that no party has an absolute majority?
I still don't see why the answer (on p. 355) is what it is. I haven't thought much about this since the start of the term, but it's still unclear.
I looked it up, and it sounds like 'integer composition' to me (though we haven't had anything about that). I haven't managed to understand why it adds up to the binomial coefficient we see on page 355.

You are right in that this is asking you to count "integer compositions" of 151 which satisfy a condition. Aigner calls compositions "ordered integer partitions".

An integer composition (ordered integer partition) of 151 is a *ordered* triple (i, j, k) such that $151 = i + j + k$. The question though wants the seats to be divided so that no party has a majority. This means we want to count compositions of 151 with the additional requirement that $i, j, k \leq 75$. Let the set of such compositions be denote by A. Let $A_i \subset A$ denote the subset of compositions whose first entry is fixed to be i. Then, $A = \sqcup_{i=1}^{75} A_i$.

Now use the summation technique, so find the size of A_i and sum. In A_i, the first entry of the composition is fixed, and so a choice of j determines the composition since, $k = 151 - i - j$. Moreover, we have $j \leq 75$ and $k = 151 - i - j \leq 75$. The last inequality gives $j \geq 76 - i$. Therefore, $76 - i \leq j \leq 75$ and hence there are only i possibilities for j and $|A_i| = i$. Then $|A| = \sum_{i=1}^{7} 5i = \binom{76}{2}$.

(3) **I do not understand theorem 5.2. Especially, I did not understand the examples, the way we should be able to apply it.**

Theorem 5.2 gives us asymptotic estimates for sequences defined by recurrences of the form $T(n) = aT(n/b) + f(n)$ and $T(1) = c$. It may not be possible to solve such recurrences explicitly, but the theorem tells us that we can still say something about the asymptotic behaviour. You are right that there are examples of recurrences that Aigner solves explicitly and we never went back and looked at them in the context of the Theorem 5.2, which appears after.

To apply the Theorem 5.2 we have to analyse the asymptotics of $f(n)$ and ask if:
(a) $f(n) \in O(n^{\log_b(a)-\epsilon})$?
(b) $f(n) \in \Theta(n^{\log_b(a)})$?
(c) $f(n) \in \Omega(n^{\log_b(a)+\epsilon})$?
Let's look at $T(n) = T(n/2) + 1$ from line (5.10) of Aigner. This recurrence has $a = 1$, $b = 2$ and $f(n) = 1$. So $\log_b(a) = 0$, and thus $f(n) = 1 \in \Theta(n^{\log_b(a)})$. Therefore, by applying Theorem 5.2 we can conclude that $T(n) \in \Theta(n^{\log_b(a)} \lg n)$ which reduces to $T(n) \in \Theta(\lg n)$. We can compare this to the solution found for the recurrence before the statement of the theorem, which was $T(n) = \lceil \lg(n) \rceil$.

Now look at $T(n) = T(n/2) + n, T(1) = 0$ from (5.12). Here $a = 1, b = 2$ and $f(n) = n$. So we have $\log_b(a) = 0$ and we fall into case 3) above. Namely, $f(n) \in \Omega(n^{\log_b(a)+\epsilon})$. To apply Theorem 5.2 we also need to check if $af(n/b) \leq cf(n)$ for some $c < 1$ and $n \geq n_0$. This is true as we can take any $0.5 < c < 1$ and we have $n/2 \leq cn$. Therefore, from the theorem we can conclude that $T(n) \in \Theta(n)$. Once again above the statement of the theorem Aigner solves $T(n) = 2n - 2$ when $n = 2^k$ and argues directly that $T(n) \in \Theta(n)$.

Can you try to apply Theorem 5.2 for the recurrence displayed in (5.13) and compare the conclusion with the one from the direct calculation?

(4) **Could you sum up the different formula linking Stirling numbers, number of partitions, etc. ?**

A good way to remember the a lot of the formulae for the Stirling numbers is to recall them to their parallels to the formulas for the binomial coefficients. First the definitions:

$\binom{n}{k}$ is the number of subsets of size k of a set of size n.
$\sigma_{n,k}$ is the number of permutations of a set of size n with k cycles.
$S_{n,k}$ is the number of partitions of a set of size n into k parts. First we have the recurrences:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$
$$\sigma_{n,k} = \sigma_{n-1,k-1} + (n-1)\sigma_{n-1,k}$$
$$S_{n,k} = S_{n-1,k-1} + kS_{n-1,k}$$

We obtained all of these recurrences by using the summation formula in different but related ways.

Then we saw the binomial theorem, which I will write just using one variable: $(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. A first Stirling number version of this is to expand the falling factorial polynomials:

$$x^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k} \sigma_{n,k}.$$

A version of this formula follows from the definition of $\sigma_{n,k}$. Remember that $n^{\underline{n}} = n!$ and by counting up permutations we get $n! = \sum_{k=0}^{n} \sigma_{n,k}$ from page 15.

We can also express x^n in terms of falling factorial polynomials:

$$x^n = \sum_{k=0}^{n} S_{n,k} x^{\underline{k}}.$$

We had already seen the above formula before when we were counting maps in formula (1.1).

These two polynomial formulas lead to Stirling inversion in Section 3.2.

I think the only other formula for Stirling numbers that we saw came from counting permutations with k cycles by summing over their types. This is on page 16 of Aigner.

$$\sigma_{n,k} = \sum_{b_1,\ldots,b_n} \frac{n!}{b_1! \ldots b_n! 1^{b_1} \ldots n^{b_n}},$$

where the sum is over $(b_1, \ldots, b_n)$ satisfying $\sum_{i=1}^{n} ib_i = n$ and $\sum_{i=1}^{n} b_i = k$.

(5) **I would love a solution to 1.13 (pdf later is fine)**

Let $f_{n,k}$ be the number of k-subsets of $\{1, \ldots, n\}$ containing no adjacent numbers. Show that

$$a) f_{n,k} = \binom{n-k+1}{k}$$

$$b) \sum_k f_{n,k} = F_{n+2}$$

Let's start with part a. Following the techniques from Section 1.2 of Aigner we will construct a bijection between

$$A = \{k - \text{subsets of } \{1, \ldots, n\} \text{ with no adjacent numbers}\}\text{and}$$

$$B = \{k - \text{subsets of } \{1, \ldots, n-k+1\}\}.$$

The set B has size $\binom{n-k+1}{k}$, so once we have the bijection we can apply the rule of equality and obtain the equality in a).

To construct a bijection, $f : A \to B$, suppose $\{a_1, a_2, \ldots, a_k\} \in A$, where the $a_i's$ are written in increasing order. Since the set has no adjacent numbers we have $a_i < a_i + 1$ for all i. Therefore,

$$1 \leq a_1 \leq a_2 - 1 \leq a_3 - 2 \leq a_4 - 3 \leq \cdots \leq a_k - k + 1 \leq n - k + 1,$$

which means that the $\{a_1, a_2 - 1, a_3 - 2, a_4 - 3, \ldots, a_k - k + 1\}$ is in B. Define

$$f(\{a_1, a_2, \ldots, a_k\}) = \{a_1, a_2 - 1, a_3 - 2, a_4 - 3, \ldots, a_k - k + 1\}.$$

(Try it out on the example: $\{1, 3, 6, 8\}$).

We can write down the inverse map $f^{-1} : B \to A$ which is

$$f^{-1}(\{b_1, \ldots, b_k\}) = \{b_1, b_2 + 1, b_3 + 2, b_4 + 3 \ldots, b_k + k - 1\}.$$

By the rule of equality we have $|A| = |B|$ and $f_{n,k} = \binom{n-k+1}{k}$.

To prove part $b)$ we show that the sums satisfy the same recurrence as the Fibonacci numbers. Firstly, check that for $n = -2$ we get $\sum_k f_{-2,k} = 0 = F_0$ and $n = -1$ we get $\sum_k f_{-1,k} = \binom{0}{0} = 1 = F_1$.

Next, consider the sum

$$\sum_k f_{n,k} + \sum_k f_{n+1,k} = \sum_{k=0}^{n} \binom{n-k+1}{k} + \sum_{k=0}^{n+1} \binom{n-k+2}{k}$$

(Some of the summands are zero above, like when $n - k + 1 < k$ but that's no big deal).

Now do a change of variables in the first sum to start the summation indexing at $k = 1$.

$$\sum_k f_{n,k} + \sum_k f_{n+1,k} = \sum_{k=1}^{n+1} \binom{n-k+2}{k-1} + \sum_{k=0}^{n+1} \binom{n-k+2}{k}$$

$$= \sum_{k=1}^{n+1} \binom{n-k+2}{k-1} + \sum_{k=1}^{n+1} \binom{n-k+2}{k} + \binom{n+2}{0}$$

Applying the recurrence relation for the binomial coefficients we obtain:

$$\sum_k f_{n,k} + \sum_k f_{n+1,k} = \sum_{k=1}^{n+1} \binom{n-k+3}{k} + \binom{n+2}{0}$$

Since $\binom{n+2}{0} = \binom{n-0+3}{0}$ we can write this as a sum from zero.

$$\sum_k f_{n,k} + \sum_k f_{n+1,k} = \sum_{k=0}^{n+1} \binom{n-k+3}{k} = \sum_k f_{n+3,k}.$$

Therefore, the sums $\sum_k f_{n,k}$ obey the same recurrence defining the Fibonacci numbers and we have $\sum_{k=0}^n f_{n,k} = F_{n+2}$.

(6) **I'm perhaps a bit late with wishes for the the session today, but here goes: Inversions is probably what I've struggled most to understand so far. Would be nice with a review of those.**

As was pointed out in the zoom chat, inversions seem like black magic. But really they are just linear algebra. Though it is always rather magical when we can import tools from various areas of mathematics that seem completely unrelated.

Strictly speaking inversions are not exactly a counting technique, but rather they give us a way of relating different sequences. But they can help us out in counting problems, since if we have two sequences related by an inversion and we have a formula for one of the sequences, inversion can provide us with a formula for the other. So maybe a good place to start practicing with inversion is to take some simple sequences and see what happens under the two main examples of inversion that we saw, binomial inversion and Stirling inversion.

Consider the sequence $u_n = 1$ for all n and let's apply binomial inversion. The formula for binomial inversion is formula (2.18) of Aigner:

$$v_n = \sum_{k=0}^n \binom{n}{k} u_k \quad \forall n \text{ if and only if } u_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} v_k \forall n.$$

So we obtain:

$$v_n = \sum_{k=0}^n \binom{n}{k} u_k = \sum_{k=0}^n \binom{n}{k} = 2^n$$

since the sum of a row of the Pingalla-Pascal triangle is 2^n. That wasn't so bad.

The statement also says that $u_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} v_k \forall n$. Which means $1 = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} 2^k$. This could be a surprising identity if we didn't already know the binomial theorem which gives a formula for $1 = (2-1)^n$.

What if you plugged in $v_n = 1$ into the binomial inversion? Then you get:

$$u_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} v_k$$

and $u_0 = 1$ and $u_n = 0$ for $n \geq 0$. So inversion is sensitive to who you call u_n and who you call v_n. This makes sense if we think about the linear algebra interpretation. If A and B are inverse matrices of each other and $Au = v$ then $Bv = u$ but Av could be anything (actually its $A^2 = u$).

Another thing to do to warm up to inversion is to find some inversion formula of your own. Any pair of basis sequences gives you an inversion formula, but it's not always so easy to find the connection coefficients! For a short review on where to use inversion see the end of the answer to the first question.

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