27 Jan 2023

MAT 2250 - Homework

1.1.

Each student must enroll 4 courses. There are 7 courses offered with maximum enrollments:

51, 30, 30, 20, 25, 12, 18.

What can we say?

Proof: Maximum # of students = 46.

Total spaces = $51+30+30+20+25+12+18 = 186$

If n is the # of students $\Rightarrow$

$4n$ = the number of enrollments.

$\Rightarrow 4n \leq 186$

$\Rightarrow n \leq \frac{186}{4}$ & n is $\in \mathbb{N}$

$\Rightarrow n \leq 46.$

1.2:

a) $S_1, \dots, S_n$ are disjoint sets. $|S_i| = a_i$

Show that the number of sets that contain at most 1 element from each S_i is $(1+a_1)\dots(1+a_n)$.

b) If $m = p_1^{a_1} \dots p_n^{a_n}$ (p_i: prime numbers), show that m has exactly $(1+a_1)\dots(1+a_n)$ divisors.

Proof:

a) Use product rule: Need to choose 1 option from each S_i, independently: either an element from S_i or none element from S_i. Hence, for each i, $\exists\ 1+a_i$ options. Product rule gives:

$(1+a_1)\dots(1+a_n)$.

b) Each divisor of $m = p_1^{a_1} \dots p_n^{a_n}$ can be written as $p_1^{b_1} \dots p_n^{b_n}$ where b_i has a_i+1 options ($b_i \in \{0, \dots, a_i\}$). Then we can apply a).

1.3. $A \subseteq \{1,\dots,100\}$, $|A| = 55$.

Show that $\exists\, a, b \in A$ such that $a - b = 9$.

(What if $|A| = 54$?) ⇒ You do at home

Proof: The idea is to consider also $B = \{1,\dots,100\} \setminus A$, & show that if A does not satisfy have 2 elements a, b such that $a - b = 9 \Rightarrow B$ has many more elements than should be.

So, we get a contradiction.

Assume that A does not have any 2 elements a & b s.t. $a - b = 9$.

Let $B = \{1,\dots,100\} \setminus A$.

If $a \in A$ & $a + 9 \leq 100 \Rightarrow a + 9 \in A$

$a + 9 \in B$.

$\Rightarrow |A \setminus \{9, \dots, 100\}| \Rightarrow 55 - 9 = 46$

We have a map

$f: A \setminus \{92,\dots,100\} \to B$

$a \longmapsto b = a + 9$

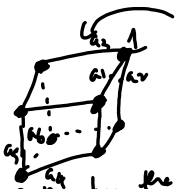
injective.

$\Rightarrow |A \setminus \{92,\dots,100\}| \leq |B| \Rightarrow |B| \geq 46$

$\not\Rightarrow$ $A \cap B = \emptyset \Rightarrow |A \cup B| = |A| + |B| \geq 55 + 46 = 101$

This is a contradiction, because: $A \cup B = \{1,\dots,100\} \Rightarrow |A \cup B| = 100$.

1.4.



12 edges

8 vertices

Can we number the 12 edges with # from 1 to 12 so that the sum of the 3 edges meeting at a vertex is the same for all 8 vertices? (e.g: $\underbrace{a_1 + a_2 + a_3}_{\text{vertex 1}} = \underbrace{a_4 + a_5 + a_6}_{\text{vertex 2}}$

Hint: Assume we can. Then what is $a_1 + a_2 + a_3$?

$\sum$ all edges $= 1+2+\dots+12 = \frac{12\times 13}{2}$

As each edge is counted for 2 corners (2 vertices) the sum of the numbers at all corners is

$$2 \times \frac{12\times 13}{2} = 12\times 13$$

Since we have 8 corners, the sum at each corner is the same, we should have that the sum at each corner is: $\frac{12\times 13}{8}$ which is not an integer, so a contradiction.

1.6. How many different words can be made from permutations of the letters in: ABRACADABRA?

Proof: $a_1\, a_2\, a_3\, a_4\, a_5\, a_6\, a_7\, a_8\, a_9\, a_{10}\, a_{11}$ Length of the word is 11

$a_1 = a_4 = a_6 = a_8 = a_{11} = A$ $\Rightarrow$ $11!$ ordered permutations.

$a_2 = a_9$ But for each of these permutations, we can swap around similar letters without creating new words. For example: There are $5!$ permutations of the 5 As. We need to divide by these. $\Rightarrow$ Answer $= \frac{11!}{5!\,2!\,1!\,1!\,2!}$

1.7. Show that $1!+2!+\dots+n!$ cannot be a square (i.e. $= m^2$ for some $m \in \mathbb{N}$) if $n > 3$.

(Rmk: $1!+2!+3! = 9 = 3^2$)

Proof: Induction??

$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$1!+2!+\dots+n!$: 1	3	9	23	143

If $n \geq 4$: the last digit of $1!+\dots+n!$ is always 3. ($n! : 10$ if $n \geq 5$).

m	0	1	2	3	4	5	6	7	8	9
$m^2 \bmod 10$	0	1	4	9	6	5	6	9	4	1

So $m^2 \bmod 10$ is never 3.

But $1!+\dots+n! \bmod 30$ is 3 when $n \geq 4$.

So $1!+\dots+n!$ cannot be m^2 for some $m \in \mathbb{N}$.