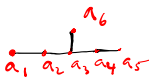
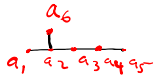
09 March 2023
MAT 2250

Examples:



G_1

Degree sequence: $(3,2,2,1,1,1)$
a_3 a_2 a_4



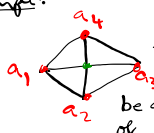
G_2

Degree sequence: $(3,2,2,1,1,1)$
a_1 a_3 a_4

Same degree sequence, but not one isomorphic. If they were isomorphic, then vertices of the same degree must be mapped to another vertex of the same degree. So we must have $\psi(a_3) = a_2$ (degree 3) ... either $a_2 \mapsto a_3$ or $a_2 \mapsto a_4$ (degree 2, 3)

Planar graph: A planar graph is a graph that can be embedded into the plane. That is, it be drawn on the plane in such a way that 2 edges only intersect at vertices.

Example: K_4:



→ This is not an embedding into the plane because the intersection point of a_1a_3 & a_2a_4 (green point) is not one of the vertex.

But we have another way to draw it:

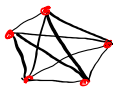


→ This is an embedding into the plane. ⇒ K_4 is planar.

Example: $K_{3,3}$ & K_5 are not planar.



$K_{3,3}$



K_5

See Planarity testing on Wikipedia.
There are fast algorithms $O(|V|)$ to test whether a graph is planar. John Bayer & Wendy Myrvold 2004.

§ 6.2 Representation of Graphs

Drawing is a way to represent graphs. It is intuitive, however, it can be difficult to read off properties of graphs from drawings, in particular if the graph is of big size. (Note: Maybe Deep learning can learn from the drawing)

Another way, easier to work with, is to use matrices.

Adjacency matrix A: $G = (V, E)$ & $n = |V|$

A is an $n \times n$ matrix,

$$A_{ij} = \begin{cases} 1 & \text{if } \exists\, ij \in E \\ 0 & \text{otherwise} \end{cases}$$

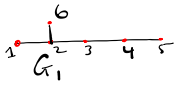
So all entries of A are 0 or 1.
Also the diagonal is 0, i.e.

$A_{ii} = 0$ $\forall i$ (because we have no loops).

Example: K_4, $n = 4$

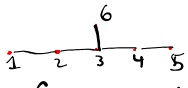
$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Example:



G_1

$$\Rightarrow A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$



G_2

$$\Rightarrow A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

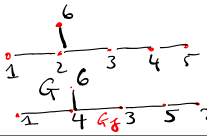
Observation: There is a 1-1 correspondence between:

Graphs on V, $|V| = n$, $V = \{1, 2, \dots, n\}$

& the set of $n \times n$ matrices whose entries are 0 or 1 & whose diagonal is 0.

Remark

If we have a permutation $f: V \to V$, then we get a new matrix A_f.



$f: V \to V$

$2 \to 4 \to 5 \to 2$

$1 \to 1, \ 3 \to 3, \ 6 \to 6$

$f = [1\ 4\ 5\ 3\ 2\ 6]$

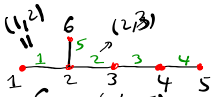
$$A_f = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Incidence matrix B: $G = (V, E)$, $|V| = n$, $|E| = q$, then B is an $n \times q$ matrix with only 1 & 0, &

$$B_{ij} = \begin{cases} 1 & \text{if } i \text{ is an endpoint of } j \\ 0 & \text{otherwise.} \end{cases}$$

vertex ↘ edge

. . .



$G = (V, E)$

$|V| = 6, \ |E| = 5$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Large graph: name an edge by uv, where u & v two end points.

Lexicographic order:

$(a_1, a_2) < (b_1, b_2)$ if either: $a_1 < b_1$ or $a_1 = b_1$ & $a_2 < b_2$

The same example above: B^T = transpose of B, so it is a 5×6 matrix.

$B^T = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$

rows of B become columns of B^T

Observation: For matrix B.

$\sum$ entries in the row i $= d(i)$

$\sum$ entries in the column j $= 2$ (each edge has 2 vertices)

$\sum$ of all rows $= \sum$ all columns

$\sum_{i \in V} d(i) = \sum_{j \in E} 2 = 2|E|$ (we found before)

Exercise: Compute BB^T for the above example

B: 6×5, B^T: 5×6

6×6 matrix

A is also a 6×6 matrix.

Q: Is there a relation between A & BB^T?

Thm: $G = (V, E)$, $|V| = n, |E| = q$

Thm: $BB^T - A = \begin{pmatrix} d(1) & & 0 \\ & \ddots & \\ 0 & & d(n) \end{pmatrix}$

Example: For the graph G above:

$BB^T - A = \begin{pmatrix} 1 & & & & & \\ & 3 & & & 0 & \\ & & 2 & 2 & & \\ & 0 & & & 1 & \\ & & & & & 1 \end{pmatrix}$

Proof: $M = BB^T$

$M_{ij} = \sum_{k \in E} B_{ik} B^T_{kj}$ $\quad i, j \in V$

By definition: $B^T_{kj} = B_{jk}$

$\Rightarrow M_{ij} = \sum_{k \in E} B_{ik} B_{jk}$

Note: $B_{ik} B_{jk} = \begin{cases} 1 & \text{if } ij = k \\ 0 & \text{otherwise} \end{cases}$

If $i \neq j$: (off the diagonal) then only at most one $k \in E$ such that $B_{ik} B_{jk} = 1$. (actually if $k = ij$)

$\Rightarrow M_{ij} = \begin{cases} 1 & \text{if } ij \in E \\ 0 & \text{if otherwise} \end{cases} = A_{ij}$

If $i = j$: (On the diagonal)

$M_{ii} = \sum_{k \in E} B_{ik} B_{ik}$

$B_{ik} B_{ik} = \begin{cases} 1 & \text{if } i \in k \\ 0 & \text{otherwise} \end{cases}$

$M_{ii} = \#$ of $k \in E : i \in k$

$= d(i)$.

& $A_{ii} = 0$

$\Rightarrow M_{ii} - A_{ii} = d(i)$. (Q.E.D.)

Matrix representation for bipartite graphs:

$G = (V = S+T, E)$, $|S| = m$, $|T| = n$, $|E| = q$.

It may be not efficient to use the $(m+n)\times(m+n)$ matrix A like before, because there will be many entries which we know are 0.

One better way is to look at $\hat{A}$ = $m \times n$ matrix (smaller size):

$$\hat{A}_{ij} = \begin{cases} 1 & \text{if } ij \in E \\ 0 & \text{otherwise.} \end{cases}$$

i ∈ S, j ∈ T

It reduces the storage need.

Example:

$|S| = 3$, $|T| = 3$, $|E| = 9$ — $K_{3,3}$

$$\Rightarrow \hat{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Why do we care about these matrices?

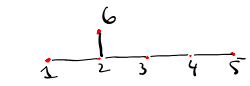
* They can be used to find whether there is a path between 2 vertices (see next).
* The eigenvalues of A can be used to rank the importance of vertices. (E.g.: Google's original search algorithm use represents each website as a vertex, & find the largest eigenvalue of A & the corresponding eigenvector.

Apply Perron-Frobenius theorem (Linear Algebra) to A: leading eigenvalue & leading eigenvector.

How to find a path of length k between vertices i & j?

Thm: Let $A_k = A^k = \underbrace{A \cdot A \cdots A}_{k \text{ times}}$.

Then $A_k[i,j]$ = # of paths of length k connecting i & j.



$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Wolfram Alpha matrix multiplication

$$A^2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 3 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

A path of length 2 from 1 → 1: 1 → 2 → 1

2 paths from 3 → 3: 3 → 2 → 3, 3 → 4 → 3

Proof of the theorem:

$$A^k = A \cdot A^{k-1}$$

So can use induction.

So proof $k = 2$ only.

$$D = A^2$$

$$D[i,j] = \sum_{k \in V} A[i,k]\,A[k,j]$$

Note $A[i,k]A[k,j] = \begin{cases} 1 & \text{iff both } A[i,k], A[k,j] = 1 \\ 0 & \text{otherwise} \end{cases}$

$A[i,k] = A[k,j] = 1$

$\Leftrightarrow ik \in E$ & $kj \in E$

$\Rightarrow i \to k \to j$ is a path of length 2 from $i \to j$.

$D[i,j]$ = # of k :
$i \to k$ & $k \to j \in E$
= # of paths of length 2 from $i \to j$. $(G.E.D.)$

Exercise: Prove the theorem for $k = 3$

$A^3 = A \cdot A^2$