13 April 2023
MAT 2250

Thm 8.5: (minimax theorem, equilibrium theorem):

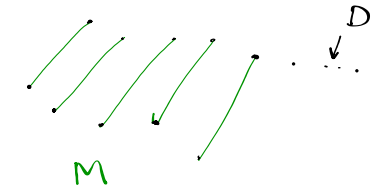
$$m(G) = \max_{M:\text{ matching of } G} |M| = \min_{D:\text{ a vertex cover } \subseteq V} |D|$$

Proof.

($\Rightarrow$) Want to show:

$$\max_M |M| \le \min_D |D|$$

M = a matching. ~~If $e \in M$,~~ $\forall u \in V$: $\Rightarrow$ $\exists$ at most one $e \in M$ so that u is a vertex of e.



($\Leftarrow$) $\max_M |M| \ge \min_D |D|$.

Want to construct a maximal matching M & construct a vertex cover D s.t.

$$\min_{D'} |D'| \le |D| \le |M| = \max_{M'} |M'|$$

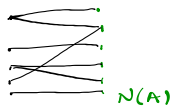
We have 2 cases: $G = (S+T, E)$

Case 1: $\forall A \subseteq S$: $|A| \le |N(A)|$

then by Thm 8.3:

$\exists$ M matching so that

$$|M| = |S|$$



Of course $D = S$ is a vertex cover (Because $G = (V = S+T, E)$ is a bipartite graph, so every edge must contain 1 vertex in S).

So $|M| = |D|$ in this case.

Case 2: $\exists$ $A \subseteq S$ such that

$$|A| > |N(A)|$$

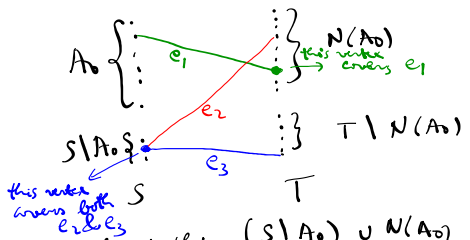
$$\delta = \max_{A \subseteq S} (|A| - |N(A)|) > 0.$$

By Theorem 8.4, $\exists$ a matching M such that $|M| = |S| - \delta$

Assume that $A_0 \subseteq S$ so that $\delta = |A_0| - |N(A_0)|$

Claim: $D = (S \setminus A_0) \cup N(A_0)$ is a vertex cover for G. We need to show that $\forall e \in G$, there is $u \in S \setminus A_0$ or $v \in N(A_0)$ such that u or v is a vertex of e.

Cases for e : an edge



So we found that $D = (S \setminus A_0) \cup N(A_0)$ covers every edge in G.

$$|D| = |S \setminus A_0| + |N(A_0)| = |S| - |A_0| + |N(A_0)| = s - s = |M|.$$

(Q.E.D.)

There are many applications of the matching problem, depending on how we deduce a question to a matching problem.

Choice of elements in a family of subsets.

Let $T = \{t_1, \dots, t_n\}$ be a set.

$A = \{A_1, \dots, A_m\}$, where $A_i \subseteq T$.

(E.g. : T = a set of people, A_i = the set of people who can also the i-th task)

Q: Can we find $t_{i_1}, \dots, t_{i_m}$ distinct elements in T so that $t_{i_j} \in A_j$ $\forall j = 1, \dots, m$?

We can deduce this to matching problem as follows. We need to construct a bipartite graph G, & so that we deduce to matchings.

$G = (V = A + T, E)$

an edge in E is $A_i t_j$ if $t_j \in A_i$.

Then a matching in G is simply a set of pairs $M = \{(t_{i_1}, A_{i_1}), \dots, (t_{i_k}, A_{i_k})\}$ where
- $t_{i_j} \in A_{i_j}$ $\forall j$
- t_{i_j} are distinct
- A_{i_j} are distinct.

So the question deduces to: Is there a matching M in G so that $|M| = |A|$? The answer is given by Theorem 8.3.

Matrices with non-negative entries:

Let $M = n \times n$ matrix $= (m_{i,j})_{i,j=1,\dots,n}$

Assume: . $m_{i,j} \geq 0$ $\forall i,j$

. $\exists\, r \in \mathbb{Z}^{>0}$ such that

$$\sum_{i=1}^{n} m_{i,j} = \sum_{i=1}^{n} m_{j,i} = r \quad \forall j.$$

($\sum_{i=1}^{n} m_{i,j}$: the sum on column j; $\sum_{i=1}^{n} m_{j,i}$: the sum on row j)

E.g. . magic squares, doubly stochastic matrix

A diagonal : is a set $\{m_{j,j_i}\}_{i=1,\dots,k}$ so that
* $m_{j,j_i} > 0$ $\forall i$
* on each $\forall j$: $\exists$ at most one i so that $j_i = j$.

(in other words, a diagonal contains positive entries, & on each row or each column there is at most one such entry)



Permutation matrix:

P = $n \times n$ matrix
- all entries are either 0 or 1.
- on each row & each column, $\exists$ exactly 1 non-zero entry.

Example: For $n = 3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

P_1 = identity matrix $\qquad P_2 \qquad P_3$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$P_1 A = A$

$$B = P_2 A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \end{pmatrix}$$

(second column; second row)

The rows in B are permutations of the rows in A.

Doubly stochastic matrix:

M = $n \times n$ matrix $= (m_{i,j})_{i,j=1,\dots,n}$
- $m_{i,j} \geq 0 \quad \forall i,j$
- $\sum_{j=1}^{n} m_{i,j} = \sum_{j=1}^{n} m_{j,i} = 1 \quad \forall i.$

$M' = rM$ has: $\sum_{j=1}^{n} m'_{i,j} = \sum_{j=1}^{n} m'_{j,i} = r \quad \forall$ $r > 0$.

Thm (Birkhoff-von Neumann): Let M be an ~~doubly stochastic matrix~~ $n \times n$ matrix with nonnegative entries, whose sum of each column & each row is the same positive number r.

Then:
1. M has a diagonal of length n. (maximal length)
2. $\exists$ positive numbers $c_1, \dots, c_t$ & permutation matrices $P_1, \dots, P_t$ so that $M = c_1 P_1 + \dots + c_t P_t$.

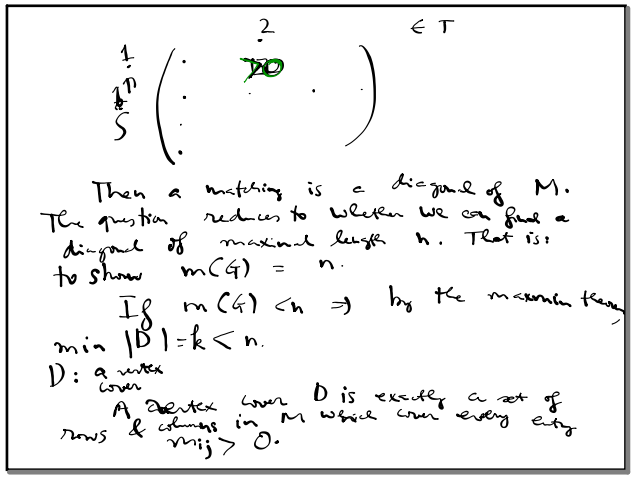
Proof: The idea is to reduce to matching problem on bipartite graphs.

$G = (V = S + T, E)$

S = the set of rows in M

T = the set of columns in M

An edge is an entry m_{ij} s.t. $m_{ij} > 0$ (So m_{ij} connects a row i & column j)



Then a matching is a diagonal of M. The question reduces to whether we can find a diagonal of maximal length n. That is: to show $m(G) = n$.

If $m(G) < n \Rightarrow$ by the max-min theorem, $\min |D| = k < n$.

D: a vertex cover

A vertex cover D is exactly a set of rows & columns in M which cover every entry $m_{ij} > 0$.

Let D be a vertex cover so that $|D| = k < n$

$$nr = \sum_{i,j=1,\dots,n} m_{ij} = \sum_{m_{ij}>0} m_{ij}$$

$$= \sum_{i \in D} m_{ij} + \sum_{j \in D} m_{ij}$$

$$= \sum_{i \in D} r + \sum_{j \in D} r$$

$$= r|D| = kr < nr$$

(contradiction).

(2) By Part 1, $\exists$ a diagonal of length n consists of $m_{1,i_1}, \dots, m_{n,i_n}$.

Now let P_1 be the permutation matrix whose 1's are at $(1, i_1), \dots, (n, i_n)$.

Let $m_1^* = \min_{j=1,\dots,n} m_{j,i_j} > 0$

Let $A_1 = A - m_1^* P_1$.

- All entries of A_1 are ≥ 0
- Σ of every row & every column are the same number $r - m_1^* \geq 0$

A_1 has the same property like A.

If $r - m_1^* = 0 \Rightarrow A_1 = 0 \Rightarrow$ done.

If not, we observe that A_1 has at least one more 0 than A.

So we can continue for A_1, at most n^2 steps. (Q.E.D.)

§ 8.2 Construction of optimal matching

We know the matching number $m(G)$ for bipartite graph $G = (V = S+T, E)$.

- Q. Can we construct a maximal matching.
- More generally, can allow weights.
- More general graphs, not just bipartite.

Matching: $G = (V, E)$. A matching is a subset $M \subseteq E$ s.t. if $e_1, e_2 \in M$, $e_1 \neq e_2 \Rightarrow e_1$ & e_2 have no common vertex.

Saturation: M = a matching. if u is a vertex of an edge in M then u is M-saturated. Otherwise, u is

M - unsaturated.

Alternating path: M a matching.

An M - alternating path is a path P of G : $P = u_0 \to u_1 \to u_2 \to \dots \to u_{2k+1}$ with
- $u_{2i} \to u_{2i+1}$ is in $E \setminus M$
- $u_{2i+1} \to u_{2i+2}$ is in M
- u_0, u_{2k+1} are M - unsaturated.

Theorem 8.9: A matching M is maximum iff there is no M - alternating path.

Proof:

($\Rightarrow$) If $\exists$ M - alternating path, then M is not maximal. We will construct a new matching M' with $|M'| > |M|$.

P: u_0 u_1 u_2 ... u_{2k} u_{2k+1}

$M' = (M \setminus$ (edges in P which are in M, i.e the green ones) $)$

$\cup$ (edges in P not in M, i.e the red ones).

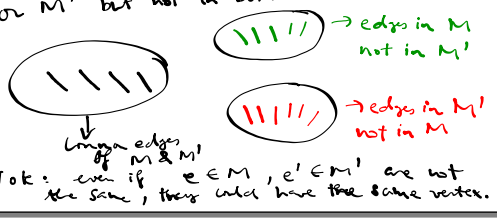
Need to show:
- $|M'| > |M|$, actually $|M'| = |M| + 1$
- M' is a matching.

Say: After taking out $u_1 \to u_2$, both u_1, u_2 become M - unsaturated. So adding $u_0 u_1$ & $u_2 u_3$ will be a matching.

($\Leftarrow$) If M is a matching which is not maximal, we will construct an M - alternating path.

M not maximal $\Rightarrow$ $\exists$ M' a matching where $|M'| = |M| + 1$.

Let $N = (M \setminus M') \cup (M' \setminus M)$ = the set of edges which are in either M or M' but not in both.

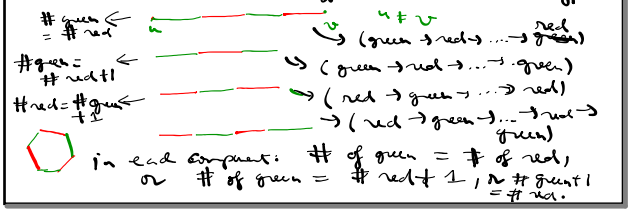


Note: even if $e \in M$, $e' \in M'$ are not the same, they could have the same vertex.

Also a vertex u can belong to at most 2 edges in N (in that case it belongs to exactly 1 in M & 1 in M').

$\Rightarrow$ Every vertex in N has degree either 1 or 2. (think about it as a graph)

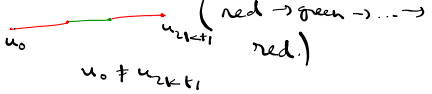
N = a graph $\Rightarrow$ N can be decomposed into components: $\in M$ (1 green) Type 1; $\in M'$ (1 red) Type 2



in each component: # of green = # of red, or # of green = # red + 1, N # green + 1 = # red.

Now $|M'| = |M| + 1$

So at least one component of N must be of the form:



$u_0 \neq u_{2k+1}$

This is an M- alternating path.