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MAT 2250

Integration by parts (Calculus)

By Fundamental Theorem of Calculus: $\int_a^b h'(x)\,dx = h(b) - h(a)$

Apply to $h(x) = f(x)g(x)$

$\Rightarrow$ & use that $h'(x) = f'(x)g(x) + f(x)g'(x)$

$\int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a)$

harder to compute

easy to compute

Integration by parts for Discrete Finite Difference

By Fundamental Theorem of Calculus for FD: $\sum_{x=a}^{b} \Delta h(x) = h(b+1) - h(a)$

What if $h(x) = f(x)g(x)$

$\Delta h(x) = h(x+1) - h(x)$

$= f(x+1)g(x+1) - f(x)g(x)$

$= [f(x+1)g(x+1) - f(x)g(x+1)] + [f(x)g(x+1) - f(x)g(x)]$

$= (f(x+1) - f(x))\,g(x+1) + f(x)\,[g(x+1) - g(x)]$

$= [\Delta f(x)]\,Eg(x) + f(x)\,\Delta g(x)$

$Eg(x) := g(x+1)$

$\sum_{x=a}^{b} \Delta f(x)\,Eg(x) + \sum_{x=a}^{b} f(x)\Delta g(x) = f(b+1)g(b+1) - f(a)g(a)$

one of these can be difficult, the other one can be easier

Example: harmonic number(s). $H_x = \sum_{m=1}^{x} \frac{1}{m}$,

$\sum_{k=1}^{n} H_k = ?$

$H_k = H_k \cdot \frac{1}{\Delta x}$ ($\Delta x = (x+1)-x$)

$f(x) = H_{x-1}$ $\quad g(x) = x$

$\sum_{x=1}^{n} f \cdot \Delta g = f(n+1)g(n+1) - f(1)g(1) - \sum_{x=1}^{n} \Delta f(x) \cdot Eg(x)$

$f(x) = H_x = \sum_{m=1}^{x} \frac{1}{m} \left(\sum_{m=1}^{x+1} \frac{1}{m}\right) - \left(\sum_{m=1}^{x} \frac{1}{m}\right)$

$\Delta f(x) = f(x+1) - f(x) = \frac{1}{x+1}$

$Eg(x) = g(x+1) = x+1$

$\Delta f(x) \cdot Eg(x) = \frac{1}{x+1} \cdot (x+1) = 1.$

Answer = $H_{(n+1)} \cdot (n+1) - H_{(1)} \cdot 1 - n$

Taylor's expansion (Calculus)

$f(x_0+\varepsilon) = f(x_0) + f'(x_0)\frac{\varepsilon}{1!} + f''(x_0)\frac{\varepsilon^2}{2!} + \cdots + \frac{f^{(n)}(x_0)}{n!}\varepsilon^n + O(\varepsilon^{n+1})$ (error term)

$D^n f$, $D = \frac{d}{dx}$

Taylor's expansion (for finite difference):

Better! (Don't have the error term!)

$\Delta \leftarrow f(x) = \sum_{k=0}^{n} \frac{\Delta^k f(0)}{k!} x^{\underline{k}}$

Polynomial of degree n Replace ε

$x_0 \to 0$, Δ, $\varepsilon^k \to x^{\underline{k}}$, $D = \frac{d}{dx}$

Proof: $k = 0, 1$ ($\Delta^0 :=$ identity map)

Taylor's expansion becomes:

$f(x) = \frac{f(0)}{0!} x^{\underline{0}} + \frac{\Delta f(0)}{1!} x^{\underline{1}}$ ($x^{\underline{0}} = 1$, $x^{\underline{1}} = x$)

$f(x)$ has degree 1 $= a + b \cdot x$

$\Rightarrow f(x) = a + bx$ (Correct!)

$f(0) = a$

$\Delta f(0) = f(1) - f(0) = (a+b) - a = b$

How to do in general: Sufficient to prove for $f(x) = x^{\underline{n}}$ & why?

Because every polynomial of degree n can be written as a linear combination of $x^{\underline{k}}$ ($0 \le k \le n$), i.e.:

$f(x) = \sum_{k=0}^{n} a_k x^{\underline{k}}$ for some constants a_k.

& the Δ^j is linear.

So only need to show that:

$\Delta^n x^{\underline{k}}(0) = \begin{cases} 0 & \text{if } 0 \le k < n \\ n! & \text{if } k = n \end{cases}$

(Use $\Delta x^{\underline{k}} = k\, x^{\underline{k-1}}$) □

Taylor's expansion for finite difference is named Newton's representation.

§2.3 Inversion

Main philosophy: Given two sequences $\{a_n\}, \{b_n\}$

Assume that we know:

$$a_n = \sum_{k=0}^{n} p_{k,n} b_k$$ (easy)

Can we find (inversion):

$$b_n = \sum_{k=0}^{n} q_{k,n} a_k$$

for (some explicitly calculated $q_{k,n}$?) difficult

A bit about ~~Linear Algebra~~:

A vector space V of finite dimension n over $\mathbb{R}$ ($\mathbb{C}$, ...) has n elements $v_1, \dots, v_n$ called a basis, which satisfies:

$v_1, \dots, v_n \in V$ are linearly independent: $\sum_{j=1}^{n} a_j v_j = 0 \Rightarrow a_1 = \dots = a_n = 0$

Every element $v \in V$ can be uniquely presented as a linear combination of $v_1, \dots, v_n$:

$$v = \sum_{j=1}^{n} a_j(v) \cdot v_j$$

$a_j(v) \in \mathbb{R}$.

If $\{v_1, \dots, v_n\}$ & $\{w_1, \dots, w_n\}$ are 2 bases, then we can write:

$$v_j = \sum_{k=1}^{n} p_{j,k} w_k$$

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also:

$$w_j = \sum_{k=1}^{n} q_{j,k} v_k .$$

What is the relations between the numbers $(p_{j,k})$ & the numbers $(q_{j,k})$?

To see this, we put them in a matrix form:

connection matrix $P = \begin{pmatrix} p_{1,1} & \cdots & p_{1,n} \\ p_{2,1} & \cdots & p_{2,n} \\ \vdots & & \\ p_{n,1} & \cdots & p_{n,n} \end{pmatrix}$ ← connection coefficients

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Then

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = P \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

Similarly, $Q = \begin{pmatrix} q_{1,1} & \cdots & q_{1,n} \\ \vdots & & \\ q_{n,1} & \cdots & q_{n,n} \end{pmatrix}$ ← connection matrix, connection coefficients

Then:

$$\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = Q \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Then:

(1) P & Q are invertible, i.e. $\det(P) \neq 0$, $\det(Q) \neq 0$.

(2) P & Q are inverse to each other, i.e. $P.Q = Id = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ (diagonal)

$$(PQ)_{n,m} = \sum_{k \geq 0} a_{n,k} b_{k,m} = [n=m] = \begin{cases} 1 & \text{if } n=m \\ 0 & \text{otherwise} \end{cases} = \delta_{n,m} \rightarrow \text{Dirac notation}$$

Now apply this to polynomials:

Let V_n = the set of all polynomials of degree $\leq n$ with coefficients in $\mathbb{R}$.

E.g.: $x+1 \in V_1$

$x^2 + 1 \notin V_1$ (degree > 1)

$x^2 - i \notin V_2$ (complex coefficient)

We know that every $f \in V_n$ can be described by $v_0(x), v_1(x), \dots, v_n(x)$, where $v_0(x) = 1$, $v_1(x) = x$, $v_n(x) = x^n$

& $v_0(x), \dots, v_n(x)$ are linearly independent.

So $\dim(V_n) = n+1$.

So $v_0(x), \dots, v_n(x)$ is a basis for V_n.

Now look at $w_0(x), w_1(x), \dots, w_n(x)$, where $w_0(x) = v_0(x-1) = 1$, $w_1(x) = v_1(x-1) = x-1$, $w_n(x) = (x-1)^n$

is also a basis.

So we can find (unique) matrices P & Q so that

$$\begin{pmatrix} v_0(x) \\ \vdots \\ v_n(x) \end{pmatrix} = P . \begin{pmatrix} w_0(x) \\ \vdots \\ w_n(x) \end{pmatrix}$$

$$\begin{pmatrix} w_0(x) \\ \vdots \\ w_n(x) \end{pmatrix} = Q . \begin{pmatrix} v_0(x) \\ \vdots \\ v_n(x) \end{pmatrix}$$

Example: $n = 2$:

$$1 = 1$$
$$x = 1(x-1) + 1$$
$$x^2 = 1(x-1)^2 + 2(x-1) + 1$$

$$\Rightarrow P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 \\ x-1 \\ (x-1)^2 \end{pmatrix} = Q . \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}$$

where $Q = P^{-1}$.

We can also find P & Q by using polynomial binomial numbers:

$$x^k = ((x-1)+1)^k = \sum_{j=0}^{k} \binom{k}{j} (x-1)^j$$

$$(x-1)^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} x^j$$

Theorem 7.2: Let $(p_0, \dots, p_n)$ & $(q_0, \dots, q_n)$ be two bases of V_n with connection coefficients $(a_{j,k})$ & $(b_{j,k})$. Then for any two sequences $u_0, u_1, u_2, \dots$ & $v_0, v_1, v_2, \dots$

$$v_j = \sum_{k=0}^{n} a_{j,k} u_k \quad \forall j \iff u_j = \sum_{k=0}^{n} b_{j,k} v_k \quad \forall j$$

Proof: $(u_0, u_1, \dots)$ is the coordinate in the basis $(p_0, p_1, \dots)$ & $(v_0, v_1, \dots)$ is the coordinate in the basis $(q_0, q_1, \dots)$. So this theorem is just telling about how to change coordinates when we change the bases. □

Example: Compute D_n the derangement number.

If $d(n,k)$ = the number of permutations of $\{1, \dots, n\}$ with exactly k fixed points.

Then

$d(n,0)$ = # with no fixed points $\Rightarrow$

$d(n,0) = D_n$. ← we want to find.

$$d(n,k) = \binom{n}{k} D_{n-k}$$

(why?: with k fixed points, π is a permutation of $n-k$ fixed points

$\pi = [a_1] \dots [a_k]\ \pi'$, choose k numbers in n numbers; π' is a derangement of $n-k$ numbers)

$$n! = \sum_{k=0}^{n} d(n,k) = \sum_{k=0}^{n} \binom{n}{k} D_{n-k} = \sum_{k=0}^{n} \binom{n}{k} D_k$$

(more symmetric)

So how do we go further?

$$n! = \sum_{k=0}^{n} \binom{n}{k} D_k \quad \forall n$$

($n!$ = v_n, D_k = u_k)

$\binom{n}{k}$ = connection coefficients for some two bases?

$\binom{n}{k}$ are actually the connection coefficients for 2 bases $(1, x-1, (x-1)^2, \dots)$ & $(1, x, x^2, \dots)$

More precisely:

$$x^n = \sum_{k=0}^{n} \binom{n}{k} (x-1)^k$$

So by the theorem, we can write:

$$D_n = \sum_{k=0}^{n} q_{n,k}\, k!$$

$q_{n,k}$: inverse connection coefficients

i.e.:

$$(x-1)^n = \sum_{k=0}^{n} q_{n,k}\, x^k$$

We found before: $q_{n,k} = (-1)^{n-k}\binom{n}{k}$ $\forall n,k$.

$$\Rightarrow D_n = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} k!$$

The point:

We want to find some sequence D_n.

Step 1: We try to find some relations like: (E_n: we knew)

$$E_n = \sum_{k=0}^{n} p_{n,k} D_k \quad \forall n$$

Step 2: We try to recognize $p_{n,k}$ as connection coefficients for some bases $(p_0, p_1, \dots)$ & $(q_0, q_1, \dots)$

Step 3: Then we find the inverse connection coefficients $q_{n,k}$.

Step 4: Then:

$$D_n = \sum_{k=0}^{n} q_{n,k} E_k \quad \forall n.$$