13 March 2023
MAT 2250

$$\lim_{n\to\infty}\frac{|a_n|}{|b_n|} = \begin{cases} 0, & \text{then } a_n < b_n \\ \infty, & \text{then } b_n < a_n \\ c, & 0<c<\infty, \text{ then } a_n = \Omega(b_n). \end{cases}$$

$n,\ n^2,\ 3^n,\ n3^n,\ \boxed{3^{n^2}} \leftarrow$

$$\lim_{n\to\infty}\frac{n3^n}{4^n} = \lim_{n\to\infty} n\left(\frac{3}{4}\right)^n = 0. \Rightarrow n3^n < 4^n.$$

Matrix representation for a bipartite graph:

Say we have $G = (V = S+T, E)$ & $|S| = n_1$, $|T| = n_2$.

If we use the usual matrix representation (adjacency matrix) A then we have a lot of 0's (for pairs of vertices in the same S or T).

Example:



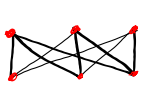
$K_{3,3}$

$$A = \begin{pmatrix} 0&0&0&1&1&1 \\ 0&0&0&1&1&1 \\ 0&0&0&1&1&1 \\ 1&1&1&0&0&0 \\ 1&1&1&0&0&0 \\ 1&1&1&0&0&0 \end{pmatrix}$$

Another way, less expensive, is to use a matrix D of size $n_1 \times n_2$ where rows represent vertices in S & columns represent vertices in T:

$$D[u,v] = \begin{cases} 1 & \text{if } uv \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

$u \in S$, $v \in T$



$K_{3,3}$

$$D = \begin{pmatrix} 1&1&1 \\ 1&1&1 \\ 1&1&1 \end{pmatrix}$$

(save storage).

Bandwidth of a graph:

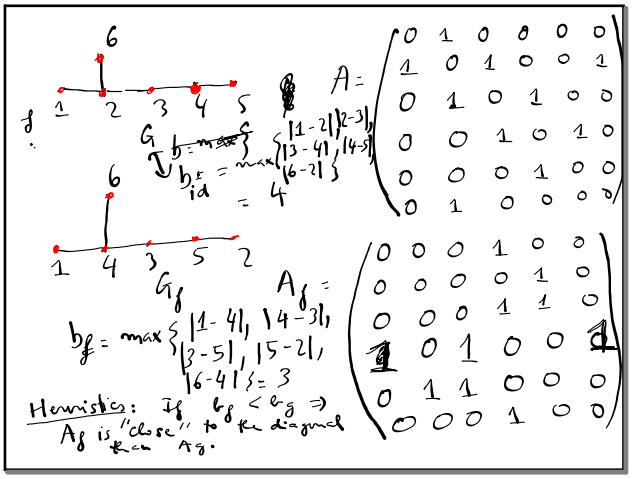
Let $G = (V, E)$ be a graph.

Let $f: V \to V$ be a permutation & G_f the induced graph (which is isomorphic to G).

$G_f \to$ adjacency matrix A_f.

$$b_f := \max_{\{uv \in E\}} |f(u) - f(v)|$$

Example:



Bandwidth of a graph:

$$b(G) = \min_{f: V \to V \text{ permutation}} b_f$$

Bandwidth pblm: . Input : a graph G
. Output $b(G)$.

Wikipedia for graph bandwidth.

Proposition: Bandwidth pblm is NP-hard.
It is also NP-hard for approximation.
(Roughly speaking, it is a hard problem.)

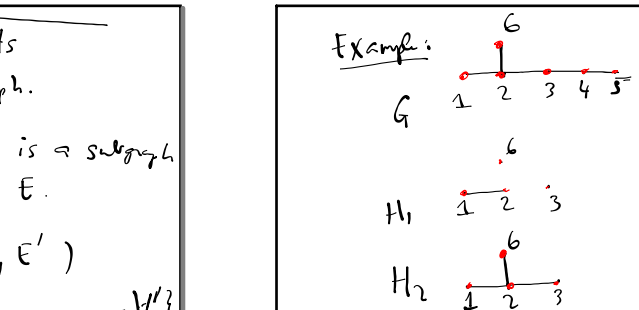
§6.3. Paths & circuits

Let $G = (V, E)$ a graph.

Subgraph: $H = (V', E')$ is a subgraph if $V' \subseteq V$ & $E' \subseteq E$.

Induced subgraph: $H = (V', E')$
where . $V' \subseteq V$
. $E' = \{uv \in E : u, v \in V'\}$.

Example:



H_1 is a subgraph of G, $V_1 = \{1,2,3,6\}$

H_2 is an induced subgraph of G, $V_2 = \{1,2,3,6\}$.

Reachability: $v \in V$ is reachable from $u \in V$ if $\exists$ a path P which starts at u & end at v.
If every v is reachable from every u, then we say that G is connected.

Example: For the graphs G, H_1, H_2 previously:
G, H_2 are connected, but H_1 is not.

Proposition: Reachability is an equivalent relation.
(Remark: it may not satisfy the reflexive requirement: that is u may be not reachable from u!)

Proof: Check: Symmetricity.
If v is reachable from u, then $\exists$ a path P: $u \to \dots \to v$.
Go back P^{-1}: $u \leftarrow \dots \leftarrow v$ shows that u is reachable from v.

Transitivity: if v is reachable from u & w is reachable from v, then $\exists$ 2 paths:
P_1: $u \to \dots \to v$
P_2: $v \to \dots \to w$
Combine P_1 & P_2: $u \to \dots \to v \to \dots \to w$
which is a path from u to w. So w is reachable from u. (Q.E.D.)

Components: For $u \in V$, let
$V(u) = \{u\} \cup \{v \in V: v$ is reachable from $u\}$.

$G(u)$ = the induced subgraph from $V(u)$.

Proposition:
1. If $v \in V(u) \Leftrightarrow u \in V(v)$.
$\Leftrightarrow V(u) = V(v)$.
2. If $u_1, u_2 \in V$, then either $G(u_1)$ is disjoint from $G(u_2)$ or $G(u_1) = G(u_2)$.

Proof: 1. If $v \in V(u) \Rightarrow v$ is reachable from $u \Rightarrow$ or $v = u \Rightarrow u$ is reachable from v or $u = v \Rightarrow u \in V(v)$.

If $u_1, u_2 \in V$ & $G(u_1) \neq G(u_2)$
$\Rightarrow V(u_1) \neq V(u_2) \Rightarrow$ by part 1):
$V(u_1) \cap V(u_2) = \phi \Rightarrow G(u_1)$ is disjoint from $G(u_2)$. (Q.E.D.)

So for each $u \in V$, we call $G(u)$ a component of G.

Property: 1) Each component of G is connected.
2) G is the disjoint union of its components.
Proof: 1) $G(u)$ is the induced subgraph with vertices $V(u)$. Let $v_1, v_2 \in$ We

Show first that if P is a path in G with u as a vertex, then P is also a path in $G(u)$. [Why? P: $u \to u_1 \to \dots \to u_m \to v$

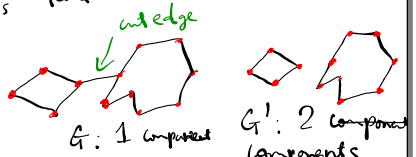
Every vertex in P is reachable from u, so they belong to $V(u)$. So $u \to u_1$, $u_1 \to u_2, \dots, u_m \to v$ are edges in $G(u)$. So $u \to u_1 \to \dots \to u_m \to v$ is a path in $G(u)$.]

2) By previous result, 2 distinct components are disjoint. It is easy to check that G is the union of its components. (Q.E.D.)

Connectedness: G is connected if it only has 1 component.

Cut edge: An edge k of a graph $G = (V, E)$ is a cut edge if the graph $G' = (V, E \setminus \{k\})$ has more components than G.

Example:



Questi

Question: $G = (V, E)$, $G' = (V, E \setminus \{k\})$

Is it true that:

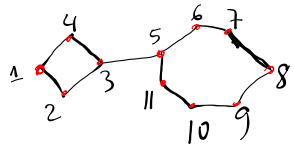
- G' always has at least the same number of components of G?
- G' always has at most 1 + number of components of G?

Distance (analogous to geodesics)

$u \neq v \in V$,

$$d(u,v) = \begin{cases} +\infty \text{ if } \exists \text{ no path from } u \text{ to } v \\ \text{min length of paths from } u \text{ to } v, \text{ otherwise} \end{cases}$$

Define $d(u,u) = 0$.



G

What is the distance from 2 to 11?

Paths: $2 \to 3 \to 5 \to 6 \to 7 \to 8 \to 9 \to 10 \to 11$: length = 8

$2 \to 1 \to 4 \to 3 \to 5 \to 11$: length = 5

$2 \to 3 \to 5 \to 11$: length = 3

Exercise: Show that $d(2, 11) = 3$.

A property of cut edge: (V, E)

If $u, v \in G = (V, E)$ & k is a cut edge of G so that u, v belongs to 2 different components of $G' = (V, E \setminus \{k\})$ then any path from u to v in G must go through the edge k.

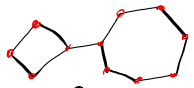
~~Exercise~~ Property: Let $G = (V, E)$ be connected. Show that the distance $d(\cdot,\cdot)$ is a metric, i.e.:

- $d(u,v) \geq 0 \ \forall u, v$, $d(u,v) = d(v,u) \ \forall u, v$
- $d(u,v) = 0$ iff $u = v$

(Triangle inequality) $d(u,v) \leq d(u,w) + d(w,v) \ \forall u, v, w$

Diameter: Let $G = (V, E)$ be connected. Then

$$\text{diam}(G) = \max_{u,v \in V} d(u,v). \quad \square$$



$\text{diam}(G) = 6$

G

How to find the diameter of a graph?

Naive way: Just choose every 2 vertices pairs then look at every paths between them, & then find the path of the least length. Quick complicated.

A more systematic way:

Let A be the adjacency matrix of G. Then if $u \neq v$:

$A^k[u,v] > 0 \iff$ there is a path of length k between u & v

$\iff d(u,v) \leq k$.

~~So, what we do is~~

Also, observe that if $|V| = n$, then $d(u,v) \leq n-1 \ \forall u, v$.

So this is the algorithm:

Step 0: A

Step 1: If A has some non-diagonal entry 0, then compute $A_1 = A^2$.

Step m: If A_{m-1} has some non-diagonal entry 0, then compute $A_m = A_{m-1}^2 = A^{2^m}$.

Because G is connected, we will stop ~~at room 6~~ for some m such that $2^m \geq n$. ($\Rightarrow m \leq \log_2 n$.)

Then: $\boxed{2^{m-1} + 1 \leq d(G) \leq 2^m}$.

We can do a little bit better & obtain the exact value for $d(G)$ with the same complexity.

Complexity of this algorithm?

We need to do m multiplications of 2 matrices of size $n \times n$: $A_0 \times A_0$, $A_1 \times A_1$, ..., $A_{m-1} \times A_{m-1}$.

So the complexity is:

$m \times$ the complexity of multiplying 2 matrices of size $n \times n$

$= O(\log_2 n) \times O(n^{2.37188\ldots})$

Conjecturally $= O(\log_2 n \times n^2)$