16 February 2023
MAT 2250

§2.4 Inclusion-Exclusion (Set theory)

(Inversion method: Linear Algebra)

→ Can be understood by Venn's diagram.

Example 1: $|A \cup B| = |A| + |B| - |A \cap B|$



Example 2:

$$|A \cup B \cup C| = (|A| + |B| + |C|) - (|A \cap B| + |B \cap C| + |C \cap A|) + |A \cap B \cap C|$$

Two ways for proof

1. Use induction + the case $A \cup B$:

$A' = A \cup B$, $B' = C$

By the case of $A \cup B$:

$|A' \cup B'| = |A'| + |B'| - |A' \cap B'|$

$A' = A \cup B \Rightarrow |A'| = |A \cup B| = |A| + |B| - |A \cap B|$

$A' \cap B' = (A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

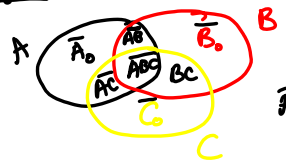
$A'' = A \cap C$, $B'' = B \cap C \Rightarrow A'' \cap B'' = (A \cap C) \cap (B \cap C) = A \cap B \cap C$

$\Rightarrow |A' \cap B'| = |A'' \cup B''|$ (use the case $A \cup B$)

$= |A''| + |B''| - |A'' \cap B''|$

$= |A \cap C| + |B \cap C| - |A \cap B \cap C|$

Way 2: Venn's diagram



$\overline{ABC} = A \cap B \cap C$

$\overline{AB} = (A \cap B) \setminus (A \cap B \cap C)$

$\bar{A}_0 = A \setminus (\overline{AB} \cup \overline{AC} \cup \overline{ABC})$

Example 3: What is $|A \cup B \cup C \cup D|$?

$\binom{4}{1} = 4 = |A| + |B| + |C| + |D|$ (intersection of 1 set(s))

$\binom{4}{2} = 6$ $- (|A \cap B| + |A \cap C| + |A \cap D| + |B \cap C| + |B \cap D| + |C \cap D|)$ (intersection of 2 sets)

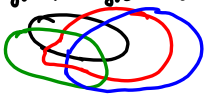
$\binom{4}{3} = 4 + (|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|)$ (intersection of 3 sets)

$\binom{4}{4} = 1$ $- (|A \cap B \cap C \cap D|)$ (intersection of 4 sets)

Proof?

. Induction & use the formula for $A \cup B$

. Venn's diagram.



Theorem: (Inclusion-Exclusion principle)
Let $B_1, \dots, B_n$ be subsets of a finite set S. Then:

① $\left|\bigcup_{i=1}^{n} B_i\right| = \sum_i |B_i| - \sum_{i<j} |B_i \cap B_j|$ (inclusion) $+ \sum_{i<j<k} |B_i \cap B_j \cap B_k| - \dots$

② $\left|S \setminus \bigcup_{i=1}^{n} B_i\right| = |S| - \sum_i |B_i|$ (exclusion) $+ \sum_{i<j} |B_i \cap B_j| - \sum_{i<j<k} |B_i \cap B_j \cap B_k| + \dots$

Proof: ① Like in the examples.

② $B_0 = S \setminus \left(\bigcup_{i=1}^{n} B_i\right)$

Then note that $B_0 \cap B_i = \emptyset$ for all $i = 1, \dots, n$.

By (1):

$$|S| = \left(|B_0| + \sum_{i=1}^{n} |B_i|\right) - \left(\sum_{1 \le i} |B_0 \cap B_i|\right) + \left(\sum_{1 \le i<j} |B_i \cap B_j|\right) + \left(\sum_{1 \le i<j} |B_0 \cap B_i \cap B_j|\right) + \sum_{1 \le i<j<k} |B_i \cap B_j \cap B_k| - \dots$$

$$\Rightarrow |B_0| = |S| - \sum_{i=1}^{n} |B_i| + \sum_{1 \le i<j} |B_i \cap B_j| - \dots \quad \square$$

Ex: Now, we can calculate D_n more systematic & with less effort.
(The inversion method is also systematic, but may need a little bit of tricks & need connection coefficients.)

$S = \{\text{all permutations of } \{1, \dots, n\}\}$
$\Rightarrow |S| = n!$

$S \supseteq D_n = \{\text{derangements} = \text{permutations with no fixed points}\}$. What is the complement of D_n?

$S \setminus D_n = \bigcup_{j=1}^{n} S_j$,

$S \supseteq S_j = \{\text{all permutations } \pi \text{ such that } \pi(j) = j\}$

By inclusion-exclusion

$$|D_n| = |S| - |S \setminus D_n| = n! - \left|\bigcup_{k=1}^{n} S_k\right|$$

$$\left|\bigcup_{k=1}^{n} S_k\right| = \sum_j |S_j| - \sum_{j<k} S_j \cap S_k + \sum_{j<k<\ell} S_j \cap S_k \cap S_\ell - \dots$$

$|S_j| = (n-1)!$ (Why: $\pi = [j] \circ \pi'$, permutation of $\{1, \dots, n\} \setminus \{j\}$, $n-1$ elements)

$|S_j \cap S_k| = (n-2)!$ (Why: $\pi = [j][k] \circ \pi''$, permutation of $\{1, \dots, n\} \setminus \{j, k\}$)

$\dots$

$$|D_n| = n! - \left[(n-1)!n - (n-2)!\binom{n}{2} + (n-3)!\binom{n}{3} - \cdots\right]$$

Summary:

Want to calculate $|D|$

Find a natural inclusion $D \subseteq S$ = bigger set

Find a good representative of the complement (exclusion):

$S \setminus D = \bigcup_{j=1}^{m} S_j$, (in particular, maybe $|S_j|$'s when S_j & S_k should be similar, $\forall\, j,k$ these intersections: $S_j \cap S_k$, $S_j \cap S_k \cap S_\ell$ ... should be easy to find.

$$|D| = |S| - |S \setminus D| = |S| - \left|\bigcup_{j=1}^{m} S_j\right|$$
$$= |S| - \sum |S_j| + \sum_{j<k} |S_j \cap S_k| - \cdots$$

(without much tricks or efforts)

One additional method:

Assume that

$$\sum_{j=0}^{n} f(x,j) = G(x,n) \quad \forall\, x \text{ & } n$$

Then

$$\sum_{j=0}^{n} \frac{\partial f}{\partial x}(x,j) = \frac{\partial G(x,n)}{\partial x}$$

$$\sum_{j=0}^{n} \int_0^x f(t,j)\,dt = \int_0^x G(t,n)\,dt.$$

Example:

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1} \quad \forall\, x \text{ & } n$$

What is $\sum_{k=1}^{n} k x^{k-1}$? $\left(\frac{\partial (x^k)}{\partial x} = k x^{k-1}\right)$

What is $\sum_{k=0}^{n} \frac{x^{k+1}}{k+1}$? $\left(\int x^k\,dx = \frac{x^{k+1}}{k+1}\right)$

Chapter 3: Generating functions

Used a lot in mathematics & physics.

§3.1: Definitions & Examples

Main idea: Say, we want to know a sequence $a_0, a_1, \ldots, a_n, \ldots$

Until now, we tried to calculate a_n directly & individually (so we find a_n independently of other a_j's).

Now the idea of the generating function is to consider all of a_n's at once, by looking at the formal power series:

$$f(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \cdots$$

Now if we can find what $f(s)$ is, without finding a_n's individually, then we have a new method to find a_n's!

Usually, this method is helpful if we have a linear recurrence among a_n's. ($f(s)$ is then usually a rational function.)

More general, if we have a sequence $a_{m,n}$, we can look at the generating function:

$$f_{\{a_{m,n}\}}(s,t) = \sum a_{m,n} s^m t^n.$$

Similarly for $a_{m,n,k}$, then we have a generating function:

$$f_{\{a_{m,n,k}\}}(s,t,u) = \sum a_{m,n,k} s^m t^n u^k$$

Example: $D_{m,n}$ = the set of permutations of $\{1,\dots,m,\dots,m+n\}$ which is of the form $\pi' \circ \pi''$, where π' is a permutation of $\{1,\dots,m\}$ & π'' is a permutation of $\{m+1,\dots,n\}$ & π has no fixed point.

Example: Stirling numbers

$S_{n,k}$, $D_{n,k}$

Calculus with formal power series:

$$A(z) = a_0 + a_1 z + a_2 z^2 + \dots$$

Define: $[z^n]\,A(z) = a_n$

(We don't care about if $A(z)$ is convergent or not.)

Sum:

$$A(z) = a_0 + a_1 z + a_2 z^2 + \dots$$
$$+\ B(z) = b_0 + b_1 z + b_2 z^2 + \dots$$
$$(a_0+b_0) + (a_1+b_1)z + (a_2+b_2)z^2 + \dots$$

$$[z^n](A(z)+B(z)) = [z^n]A(z) + [z^n]B(z).$$

Product: Multiply $A(z)\,B(z)$ formally, by & group those with the same z^n:

$$\left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \dots\right)$$
$$\times \left(b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + \dots\right)$$
$$= \left(a_0 b_0 + a_0 b_1 z + a_0 b_2 z^2 + a_0 b_3 z^3 + a_0 b_4 z^4 + \dots\right)$$
$$+ \left(a_1 b_0 z + a_1 b_1 z^2 + a_1 b_2 z^3 + a_1 b_3 z^4 + \dots\right)$$
$$+ \left(a_2 b_0 z^2 + a_2 b_1 z^3 + a_2 b_2 z^4 + \dots\right)$$
$$+ (\dots\dots)$$

$$a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \dots$$

So:

$$[z^n]\left(A(z)B(z)\right) = \sum_{i=0}^{n} \left([z^i]\,A(z)\right) \times \left([z^{n-i}]\,B(z)\right)$$

Proposition (inverse)

If $[z^0]\,A(z) \neq 0$ then there is $B(z)$ so that $A(z)B(z) = 1$.

Proof: Let $B(z) = b_0 + b_1 z + b_2 z^2 + \dots$

Want to show that if $a_0 \neq 0$, then we can find $b_0, b_1, \dots$ such that

$$A(z)B(z) = 1 = 1 + 0z + 0z^2 + 0z^3 + \dots$$

By the formula for product:

$a_0 b_0 = 1 \Rightarrow b_0 = 1/a_0$ (OK, because $a_0 \neq 0$)

$a_0 b_1 + a_1 b_0 = 0 \Rightarrow a_0 b_1 = -a_1 b_0$ (OK, because $a_0 \neq 0$)

$a_0 b_2 + a_1 b_1 + a_2 b_0 = 0 \Rightarrow a_0 b_2 = -a_1 b_1 - a_2 b_0$ (OK, because $a_0 \neq 0$)

$\dots$

So we can prove by induction that b_n can be found. □

If $[z^0]A(z) \neq 0$ & $A(z)B(z) = 1$, then we write: $B(z) \stackrel{\text{def}}{=} \frac{1}{A(z)}$.

Then $\frac{C(z)}{A(z)} \stackrel{\text{def}}{=} C(z)B(z)$.

Some properties:

$A(z)B(z) = B(z)A(z)$ (commutative)

$(A(z)B(z))C(z) = A(z)(B(z)C(z))$ (associative)

$A(z)(B(z)+C(z)) = A(z)B(z) + A(z)C(z)$.

These can be proved by observing the following:

Lemma: Let $A(z) = a_0 + a_1 z + a_2 z^2 + \dots$
$B(z) = b_0 + b_1 z + b_2 z^2 + \dots$

$A_k(z) :=$ cut off at z^k
$= a_0 + a_1 z + \dots + a_k z^k$

$B_k(z) :=$ cut off at z^k
$= b_0 + b_1 z + \dots + b_k z^k$

Then: $[z^k](A(z)B(z)) = [z^k](A_k(z)B_k(z))$ (polynomials)

& $A_k(z), B_k(z)$ are polynomials, so:

$A_k(z)B_k(z) = B_k(z)A_k(z)$.

↪ this is the usual multiplication of polynomials = the same if we regard them as formal power series.

Exercise: Let $A(z)$ & $B(z)$ be polynomials

Show that $A(z)B(z)$ (as polynomials) $=$ $A(z)B(z)$ as formal power series