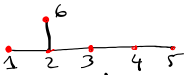
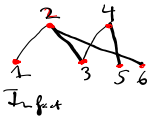
16 March 2023
MAT 2250

Theorem: (Allowing to check if a graph is bipartite) A graph G with $n \geq 2$ vertices is bipartite iff all circuits have even length. In particular, if G has no circuit at all, then G is bipartite.

Example:



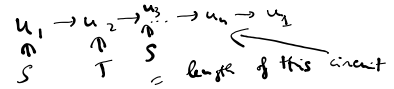
G has no circuit
$\Rightarrow$ G is bipartite.



Infact

Proof:

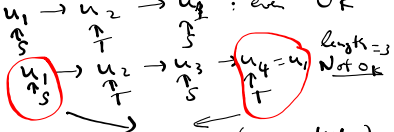
($\Rightarrow$) Assume that G is a bipartite graph. Consider a circuit: " $(V = S \dot{+} T, E)$



If = length of this circuit

Claim: n is even

Example:



$u_1 \in S \cap T$ (contradiction)

($\Leftarrow$) Assume that every circuit in G has even length.
Then: we fix $u \in V$.

Define:

$S = \{ v \in V : \ d(u,v) = \text{even} \}$

$T = \{ v \in V : \ d(u,v) = \text{odd} \}$.

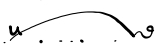
We claim that: $u \in S$ ($d(u,u) = 0$ even)

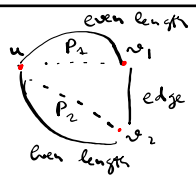
$S \cap T = \phi$ (easy)

If An edge in E must contain a vertex in S & a vertex in T.

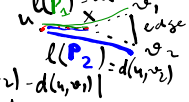
Now let $v_1 v_2$ be an edge in E.

We need to show that we cannot have both $v_1, v_2 \in S$ or both $v_1, v_2 \in T$. For example, assume that $v_1, v_2 \in S$, then we will obtain a contradiction.





Now $P_1 \cap P_2 \supseteq \{u\}$. Let x be the last common vertex of P_1 & P_2.



$|d(u,v_2) - d(u,v_1)| \leq 1$

$\Rightarrow d(u,v_1) = d(u,v_2)$.

$d(u,v_2) \leq d(u,v_1) + d(v_1,v_2)$

& both $d(u,v_1)$, $d(u,v_2)$ are even numbers

Then by triangle inequality:

$d(x,v_1) = \ell(P_1) - d(u,x) = d(x,v_2) = d(u,v_1)$

So the length of the path from $x \to v_1$ on P_1 is the same as the length of the path from $x \to v_2$ on P_2.

Then $x \to v_1 \to v_2 \to x$ is circuit with length = length (x, v_1) + $d(v_1, v_2)$ (= 1) + length (x, v_2)

$= 2\, d(x, v_1) + 1$ = odd number, a contradiction. (Q.E.D.)

An algorithm to check if a graph G is bipartite:

Choose $u_1 \in G$, $S = \{u_1\}$, $T = \emptyset$

$\Rightarrow V_1 = V(u_1)$ = neighbors of u_1

$\Rightarrow S = S \cup V(u_1)$.

V_2 = neighbors of a point in S.

If $V_2 \cap S = \emptyset \Rightarrow G$ is not bipartite

Otherwise:

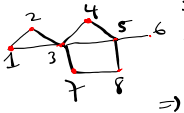
$T = T \cup V_2$

V_3 = neighbors of points in T

If $V_3 \cap T \neq \emptyset \Rightarrow G$ is not bipartite

Otherwise: $S = S \cup V_3$

Repeat until we find that G is not bipartite or we exhaust all vertices in G.



Step 1: $S = \{1\}$, $T = \emptyset$

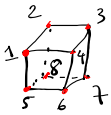
Step 2: $V(1) = \{2, 3\}$

$V(1) \cap S = \emptyset$

$\Rightarrow T = T \cup V(1) = \{2, 3\}$

Step 3: V_2 = neighbors of T $= V(2) \cup V(3) = \{1, 3\} \cup \{1, 2, 4, 5\}$

$V_2 \cap T \neq \emptyset \Rightarrow T$ is not bipartite. 7 8



Step 1: $S = \{1\}$, $T = \emptyset$

Step 2: V_1 = neighbors of S $= V(1) = \{2, 5, 4\}$

$V_1 \cap S = \emptyset \Rightarrow$ Ok

$T = T \cup V_1 = \{2, 5, 4\}$

Step 3: V_2 = neighbors of T $= V(2) \cup V(5) \cup V(4) = \{1, 8, 3\} \cup \{1, 8, 6\} \cup \{1, 3, 6\} = \{1, 3, 6, 8\}$.

$V_2 \cap T = \emptyset \Rightarrow$ Ok

$S = S \cup V_2 = \{1, 3, 6, 8\}$.

Step 4: V_3 = neighbors of S $= V(1) \cup V(3) \cup V(6) \cup V(8) = \{2, 4, 5, 7\}$. Ok. $T = T \cup V_3 = \{2, 4, 5, 7\}$

$V_3 \cap S = \emptyset \Rightarrow$ Ok.

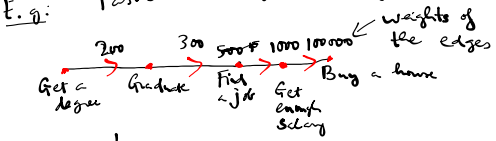
Now we got every vertices in $G \Rightarrow G$ is bipartite.

§6.4 Directed graphs

In many problems, it is useful to allow some one-way edges as well.

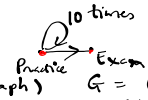
E.g.: 1-way Roads

E.g.: Tasks with different priority



E.g. Loop

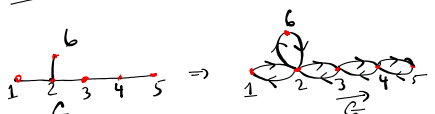
Coding in computers



Directed graphs. (Di-graph) $G = (V, E)$ but now E is a subset of $V \times V$ (and not 2-subset of V.)

We assume that $(u,u) \notin E$.
Each edge appears at most 1.

Example: Undirected graph is a directed graph.



Each undirected edge $\Leftrightarrow$ 2 directed edges

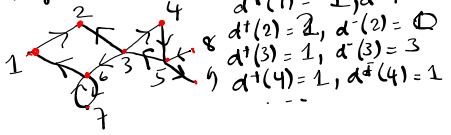
Source: u is a source of an edge k, if the edge is like this: $u \rightarrow$ We think $u = k^-$

Sink: u is a sink of an edge k if the edge k is like this: $\rightarrow u$ $u = k^+$.

- In degree:
$d^+(u) = \#\{k \in E : u = k^+\}$
$= \#$ of edges having u as a sink (coming to u)
- Out degree:
$d^-(u) = \#\{k \in E : u = k^-\}$
$= \#$ of edges leaving u as a source (go away from u)

Observation: If $\vec{G}$ is undirected $\Rightarrow d^+(u) = d^-(u)$.

Proposition: $\sum_{u \in V} d^+(u) = \sum_{u \in V} d^-(u) = |E|$,
counted as if ~~$\vec{G}$ is undirected~~.



$d^+(1) = 1, d^-(1) = 1$
$d^+(2) = 2, d^-(2) = 0$
$d^+(3) = 1, d^-(3) = 3$
$d^+(4) = 1, d^-(4) = 1$
...

Proof: Each edge E has a source & a sink. So each edge E counts 1 for $\sum_u d^+(u)$ & also 1 for $\sum_u d^-(u)$. (Q.E.D.)

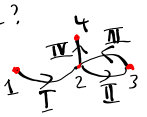
In particular, for undirected graphs.
$d^+(u) = d^-(u)$ $\forall u$
$d(u) = d^+(u) + d^-(u)$
$\Rightarrow \sum_u d(u) = 2|E|$.

Incidence matrix $\vec{B}$: $\vec{G} = (V, E)$, $|V| = n$,
$|E| = q \Rightarrow$ $\vec{B}$ has an $n \times q$ matrix,
also it has 3 values $\{+1, -1, 0\}$:

$$\vec{B}[u,k] = \begin{cases} 1 & \text{if } u = k^- \\ -1 & \text{if } u = k^+ \\ 0 & \text{if } u \notin k. \end{cases}$$

(For undirected graphs we have:
$BB^T = A + \begin{pmatrix} d(u_1) & & 0 \\ & \ddots & \\ 0 & & d(u_n) \end{pmatrix}$)

In the setting of directed graphs, how can we generalize?



$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & +1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Observe: $\sum$ of entries on each column $= 0$.

$$B^T = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

Use Wolfram Alpha to calculate BB^T.

$$B B^T = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 4 & -2 & -1 \\ 0 & -2 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ & 4 & & \\ & & 2 & \\ & & & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

diagonal D $\qquad$ A

$D^{[u,u]}$ = the degree of the vertex u if we think the edges as undirected:

$A^{[u,v]}$ = # $\{k \in E : u = k^+ \text{ or } u = k^-\}$

= # of edges connecting u & v, we don't care about direction.

Theorem: if $\overrightarrow{G} = (V, \overrightarrow{E})$ directed graphs

$\overrightarrow{B}$ = incidence matrix, then:

$$\overrightarrow{B} \cdot \overrightarrow{B}^T = D - A$$

D = diagonal matrix, $D[u,u] =$ # $\{k \in E : u = k^+ \text{ or } u = k^-\}$

A = off-diagonal matrix $A[u,v] =$ # $\{k \in E : u = k^+ \,\&\, v = k^- \text{ or } u = k^- \,\&\, v = k^+\}$

Proof: like in the undirected case. (Q.E.D.)

Directed path: $u_1 \to \dots \to u_n$

Directed circuit: $u_1 \to \dots \to u_n \to u_1$

Acyclic graph: A directed graph with no directed circuit. (Circuit = cycle = cyclic)

Proposition: If $\overrightarrow{G}$ is an acyclic graph, then $\exists\, u \in V$ such that $d^+(u) = 0$.

Similarly, $\exists\, v \in V$ such that $d^-(v) = 0$.

Proof: Assume otherwise. For example, $\forall u \in V : d^+(u) \neq 0$.

Meaning for every $u \in V$, $\exists\, v \in V$ & an edge $u \leftarrow v$.

Choose $u_1 \in V$. We find $u_1 \leftarrow u_2$.

If $u_2 = u_1$, we have a circuit, contradiction.

So $u_1 \neq u_2$ & we can find: $u_1 \leftarrow u_2 \leftarrow u_3$.

If $u_3 \neq u_1$, $u_3 \neq u_2$ then we have a circuit. Otherwise, we find: $u_1 \leftarrow u_2 \leftarrow u_3 \leftarrow u_4$ & $u_4 \neq u_1, u_2, u_3$.

We can keep going like this:

$$u_1 \leftarrow u_2 \leftarrow \dots \leftarrow u_N,$$

$$u_i \neq u_j \quad \forall\, i \neq j$$

& $\forall N$.

However, this is a contradiction because $|V| < \infty$, so if $N > |V|$ we must have some $i \neq j$ so that $u_i = u_j$ & we have a circuit, a contradiction. (Q.E.D.)

Maze problem:



Algorithm:

Step 0: Choose any $u_0 \in V$. Start at u_0, go to every point in V, & then come back to u_0.

Rules:

(1) For each edge, we go only 1 time for

(2)