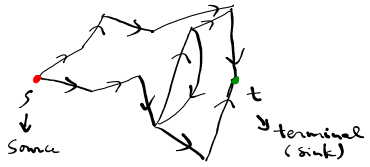
20 April 2023
MAT 2250

§ 8.3 Flows in networks



Def (Network): Consists of
- A directed graph $G = (V, E)$
- A weight function on the edges:
  $c: E \to \mathbb{N}_{\geq 0}$ (c = capacity)
- $s, t \in V$: source & terminal vertices.

Def (Flow): A flow is a function:
$f: E \to \mathbb{N}_{\geq 0}$

Def (Net flow):
$$x \in V: \quad \partial f(x) := \sum_{\substack{k \in E \\ k^+ = x}} f(k) - \sum_{\substack{k \in E \\ k^- = x}} f(k)$$

$\partial f(x)$
$= (5+7+0) \Leftarrow$ 
$- (3+3+1+2) = 3$

Net flow = (flow on green ones) − (flow on red ones)

Def (Admissible flow): Need 2 conditions:
- $0 \leq f(k) \leq c(k) \quad \forall k \in E$
- $\partial f(x) = 0$ if $x \neq s, t$

Lemma: f any flow.

① $\sum_{x \in V} \partial f(x) = 0$

② If f is admissible, then:
$\partial f(s) = -\partial f(t)$.

Proof:

① If k an edge, then in $\sum_{x \in V} \partial f(x)$, we count
$+f(k)$ for k^+
$-f(k)$ for k^-
$= 0$.
$\Rightarrow$ we are done.

② If f is admissible $\Rightarrow \partial f(x) = 0 \ \forall x \neq s, t$.
$\Rightarrow 0 = \sum_{x \in V} \partial f(x) = \partial f(s) + \partial f(t)$.

Def (Value of an admissible flow):
$$w(f) := \partial f(t).$$
$$(= -\partial f(s)).$$

Remark: In the literature, there is the more general definition of net flow, which is for 2 vertices u, v we define $F(u,v)$ with the property that $F(u,v) = -F(v,u)$.

One way to define (for an admissible flow & $u, v \neq s, t$):
$$F(u,v) = \sum_{k^+ = v} f(k) - \sum_{k^- = u} f(k)$$
(if $u = v \Rightarrow F(u,v) = \partial f(u)$).

Question: (Maximal flow): Find an admissible flow f with maximal value $w(f)$?

Def: (s-t cut): A cut of a network is simply a partition $V = X \amalg Y$.
An s-t cut is a cut with $s \in X$, $t \in Y$.
Capacity of the cut is:
$$c(X,Y) = \sum_{\substack{k \in E \\ k^- \in X, \\ k^+ \in Y}} c(k)$$
(i.e.: k starts from X & ends in Y)

Rmk: What we do with a cut is to construct a directed bipartite network.

The next lemma gives a first indication of the relation between bipartite graphs & the maximal flow problem.

Lemma 8.16: If f is admissible flow & (X,Y) is an s-t cut, then:
$$w(f) \leq c(X,Y).$$

Proof:
Let (X,Y) be an s-t cut.
Then $t \in Y$, & $s \notin Y$.
f is admissible $\Rightarrow \partial f(z) = 0 \ \forall z \neq s,t$.
$$w(f) = \partial f(t) = \sum_{y \in Y} \partial f(y).$$
Look at an edge k. If $+f(k)$ or $-f(k)$ appears in the RHS, then $k^+ \in Y$ or $k^- \in Y$.

If both k^+ & k^- are in Y then the total contribution of k in $\sum_{y \in Y} \partial f(y)$ is 0.
$+f(k)$ for k^+ & $-f(k)$ for k^-.
So we only care about the edges k which have one vertex in Y & one vertex in X.
If $k^+ = y \in Y$ (so $k^- = x \in X$), then we have a contribution of $+f(k)$ in $\partial f(y)$.
If $k^- = y \in Y$ (so $k^+ = x \in X$), then we have a contribution of $-f(k)$ in $\partial f(y)$.
So:
$$w(f) = \sum_{y \in Y} \partial f(y) = \sum_{\substack{k: \\ k^+ \in Y, \\ k^- \in X}} f(k) - \sum_{\substack{k: \\ k^- \in Y \\ k^+ \in X}} f(k)$$

$$\leq \sum_{\substack{k: \\ k^+ \in Y \\ k^- \in X}} f(k)$$
$$= c(X,Y). \qquad \text{Q.E.D.}$$

Rmk: In the above, the estimates are very rough. So the next theorem is quite surprising.

Def (Augmenting path: **Note:** in the text book it is called increasing path & its description is not very clear)
It is a path in the associated undirected graph from the source s to the terminal t
$$P = u_0 = s, u_1, u_2, \dots, u_{k-1}, u_k = t,$$

which satisfies the following conditions:

(By definition of the associated undirected graph, either $u_i u_{i+1}$ or $u_{i+1} u_i$ (on book) is an edge in the original graph G.)

If $k_{i,+} = u_i u_{i+1}$ is an edge in G (& $u_{i+1} u_i$ is not an edge in G) – called a forward edge – then we want: $f(k_{i,+}) < c(k_{i,+})$.

If $k_{i,-} = u_i u_{i-1}$ is an edge in G (& $u_{i-1} u_i$ is not an edge in G) – called a backward edge – then we want $f(k_{i,-}) > 0$.

If both $u_i u_{i-1}$ & $u_{i-1} u_i$ are edges in $G \Rightarrow$ you are free to choose one, provided the above conditions are satisfied.

Reason is here:

We will add a positive amount to forward edges & subtract a positive amount to backward edges.

Example on the web.

Def (Partition associated to an admissible flow):

Define:

$X_f = \{ x \in V : x = s$ or there is an augmenting path from s to $x \}$

$Y_f = V \setminus X_f$.

Rmk: $s \in X_f$. But we don't know whether $t \in Y_f$.

Thm 8.17 (Ford & Fulkerson: maximal flow – minimal cut)

G = a network.

f : admissible flow

TFAE:

a) f is a maximal flow

b) There is no augmenting path from s to t.

c) (X_f, Y_f) is an s-t cut. (i.e. $t \in Y_f$).

In all of these cases: $w(f) = c(X_f, Y_f)$.

Proof:

$a \Rightarrow b$: Let P be an augmenting path from s to t. $P = u_0 = s, u_1, \dots, u_k = t$

Define

$$f_P(k) = \begin{cases} 1 & \text{if } k \text{ is a forward edge in } P \\ -1 & \text{if } k \text{ is a backward edge in } P \\ 0 & \text{if } k \text{ is not an edge in } P. \end{cases}$$

$$\alpha = \min \{ c(k) - f(k) : k \text{ forward edge}, \; f(k) : k \text{ backward} \} > 0$$

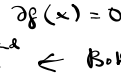
$g = f + \alpha f_P$

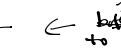
Ch. α = integer $\Rightarrow$ g also has integer values

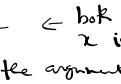
$g \geq 0$: Consider $k \in P$ & $k \notin P$.

g is still an admissible flow:

need to check $\partial g(x) = 0 \quad \forall x \in V(P)$

Net flow $= 0 \leftarrow$  $\leftarrow$ Both edges are forward

Net flow $= 0 \leftarrow$  $\leftarrow$ both edges incident to x is forward & 1 is backward.

Net flow $= 0 \leftarrow$  $\leftarrow$ both edges incident to x in P are backward

(Look at the argument for the example on the web).

So just look at the net flow of the actual path corresponding to P in G, we have 3 cases

$$w(g) = w(f) + \alpha\, \partial g_P(t) = w(f) + \alpha > w(f).$$

$\Rightarrow f$ is not maximal flow.

$b \Rightarrow c$: if there is no augmenting path from s to $t \Rightarrow t \notin X_f \Rightarrow t \in Y_f$

$\Rightarrow (X_f, Y_f)$ is an s-t cut.

$c \Rightarrow a$:

First show that $\forall$ edges k with $k^- = x \in X_f$ & $k^+ = y \in Y_f$ then $f(k) = c(k)$.

if not: Let $s \to x$ be an augmenting path

Then we can add: $s \to x \to y \Rightarrow y \in X_f$ (contradiction)

Similarly, if $k^- = y \in Y_f$, $k^+ = x \in X_f$

$\Rightarrow f(k) = 0$.

Then by proof of Lemma 8.16: (1) (0)

$$w(f) = \sum_{\substack{k^- \in X_f \\ k^+ \in Y_f}} f(k) - \sum_{\substack{k^+ \in X_f \\ k^- \in Y_f}} f(k)$$

$$= c(X_f, Y_f).$$

And we know by Lemma 8.16:

$\forall$ g admissible flow: $w(g) \le c(X_f, Y_f) \Rightarrow f$ is maximal (Q.E.D.)

Corollary 8.18: A maximal flow exists.

Proof: Start from $f = 0$.

If f has augmenting path $\Rightarrow$ can modify to f_1 with $w(f_1) > w(f) + 1$.

(From the proof of $a \Rightarrow b$ in Thm 8.17)

Now

$$w(f_1) \le \sum_{k \in E} c(k)$$

$\Rightarrow$ So we will terminate after a finite time. (Q.E.D.)

Algorithm (Ford Fulkerson): $G = (V, E)$ network with integer values

Initialization flow $f = 0$

Step 1: While G, f has an augmenting path

$\Rightarrow$ Do modification like in the proof of $a \Rightarrow b$ in Thm 8.17.

By Cor 8.18, this algorithm terminates after a finite time.

How to code the task of finding an augmenting path?

Residual graph: $G_f = (V, E_f)$

For each edge $k \in E$ (from the original G) we add a new edge k_{rev} & define the weights:

forward edge $\leftarrow w_{G_f}(k) = c(k) - f(k)$

backward edge $\leftarrow w_{G_f}(k_{rev}) = f(k)$.

Then keep only edges in G_f for which the weight is > 0.

An augmenting path in G is exactly a directed path from s to t in G_f.