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MAT 2250

§3.1 (cont.)

One common way to get formal power series is to use Newton's expansion in Calculus.

(Under some conditions, e.g. $f(z)$ is real analytic in an interval around 0), then:

$$f(z) = f(0) + \frac{f(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \cdots$$

Example.

$f(z) = \frac{1}{1-z}$, real analytic around 0.

$f(0) = 1$,

$f'(z) = \frac{1}{(1-z)^2} \Rightarrow f'(0) = 1$

$f''(z) = \frac{2}{(1-z)^3} \Rightarrow f''(0) = 2 = 2!$

$f'''(z) = \frac{2\cdot 3}{(1-z)^4} \Rightarrow f'''(0) = 3!$

$\vdots$

By induction: $f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$

$\Rightarrow f^{(n)}(0) = n!$

By Taylor's expansion:

$$\frac{1}{1-z} = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \cdots$$
$$= 1 + \frac{1!}{1!}z + \frac{2!}{2!}z^2 + \cdots$$
$$= 1 + z + z^2 + z^3 + \cdots$$

(We knew by elementary methods that $1 + z + \cdots + z^n = \frac{1-z^{n+1}}{1-z}$, taking $n \to \infty$ & for $|z|<1$ we get $1 + z + \cdots + z^n + \cdots = \frac{1}{1-z}$.)

Another method is to use the polynomial method:

If we knew that

$$P(c) = Q(c) \quad \forall c \in \mathbb{N}$$

& $P(c)$, $Q(c)$ are polynomials whose $c \in \mathbb{N}$ then we can extend to $c \in \mathbb{C}$ & prove by using Taylor's expansion.

Example: $\sum_{n\geq 0} \binom{c}{n} z^n = (1+z)^c$, $c \in \mathbb{C}$.

When $c \in \mathbb{N}$, this is just the binomial theorem. For $c \in \mathbb{C}$, we need to explain what is $(1+z)^c$.

If $z \in \mathbb{R}$, $|z| < 1$

$\Rightarrow 1 + z > 0$

$e^{\log(1+z)} = 1 + z$ for $c \in \mathbb{C}$

We define: $(1+z)^c := e^{c\log(1+z)}$

Put $w = c\log(1+z)$, we define (by using Taylor's expansion for e^w, with $w \in \mathbb{R}$, $w \in \mathbb{C}$):

$e^{c\log(1+z)} = e^w := 1 + \frac{w}{1!} + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots$

Then: $f(z) = (1+z)^c = e^{c\log(1)} = e^{c\cdot 0} = e^0 = 1$

So $f(0) = 1^c = e^{c\log(1)}$

$f(z) = (1+z)^c$

$f'(z) = ?$ $\quad c(1+z)^{c-1}$

$f'(0) = c$

(Be careful about if $c \in \mathbb{N}$!)

HW: What is $f''(z)$? $f''(0)$? $f'''(z)$? $f'''(0)$?

Some properties:

Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ be a formal power series.

Under some mild conditions (e.g. $A(z)$ is analytic around 0), then:

$\sum_{n=0}^{\infty} n a_n z^n = z A'(z)$

$\sum_{n=0}^{\infty} \frac{1}{n+1} a_n z^n = \frac{1}{z} \int_0^z A(t)\,dt.$

Proof: $A(z) = \sum_{n=0}^{\infty} a_n z^n$

Taking derivative w.r.t. z:

On the LHS we get $A'(z)$

On the RHS we can do derivative term by term: $\left(\sum_{n=0}^{\infty} a_n z^n\right)' = \sum_{n=0}^{\infty} (a_n z^n)'$

$= \sum_{n=0}^{\infty} n a_n z^{n-1}$

Multiplying both sides with z, we get:

$z A'(z) = z\left(\sum n a_n z^{n-1}\right)$

$= \sum n a_n z^n.$

(FTC) Taking integration on both sides:

$\int_0^z A(t)\,dt = \int_0^z \left(\sum_{n\geq 0} a_n t^n\right) dt$

$= \sum_{n\geq 0} \int_0^z a_n t^n\,dt = \sum_{n\geq 0} \frac{a_n z^{n+1}}{n+1}$

Dividing both sides by z:

$\frac{1}{z}\int_0^z A(t)\,dt = \sum_{n\geq 0} \frac{a_n z^n}{n+1}.$ □

Example:

$A(z) = \log(1+z)$

We know (by formula h on page 66):

$A(z) = \sum_{n\geq 1} \frac{(-1)^{n+1} z^n}{n}$

$a_n = \frac{(-1)^{n+1}}{n} \Rightarrow a_n \cdot n = (-1)^{n+1}$

$\Rightarrow \sum_{n\geq 1} (-1)^{n+1} z^n = z \cdot A'(z) = \frac{z}{1+z}$.

$\frac{a_n}{n+1} = \frac{(-1)^{n+1}}{n(n+1)}$

HW: What is $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n(n+1)} z^n$?

Example:

$\frac{1}{1-cz} = \sum_{n=0}^{\infty} (cz)^n$

What is $\frac{1}{(1-cz)^2}$?

In the book (page 67) they have a way to do it. Here is another way:

$A(z) = \frac{1}{1-cz} \Rightarrow A'(z) = \frac{c}{(1-cz)^2}$

$\Rightarrow \frac{c}{(1-cz)^2} = \sum_{n\geq 0} ((cz)^n)' = \sum_{n\geq 0} (cz)^{n-1} \cdot n$ □

Power series for rational functions:

A rational function is a function of the form $f(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ & $Q(z)$ are polynomials.

There is a general method to get power series for rational functions.

Step 1: We can write $f(z) = f_1(z) + \frac{P_1(z)}{Q_1(z)}$, where $f_1(z), P_1(z)$ & $Q_1(z)$ are polynomials, & $\deg P_1 < \deg Q_1$. (Euclidean division algorithm)

$f(z) = \frac{z^3}{z^2-2z+1} = \frac{P(z)}{Q(z)}$

$3 = \deg P > \deg Q = 2$

So we divide P by Q:

$z^3 = z(z^2-2z+1) + 2z^2 - z$

$= z(z^2-2z+1) + 2(z^2-2z+1) + 4z - 2 - z$

$= (z+2)(z^2-2z+1) + 3z - 2$

So $f(z) = \underbrace{f_1(z)}_{z+2} + \frac{\overbrace{3z-2}^{P_1(z)}}{\underbrace{z^2-2z+1}_{Q_1(z)}}$

Step 2: We use of Fundamental theorem of Algebra:

If $Q_1(z) \in \mathbb{C}[z]$ is a non-constant polynomial of degree n, then $\exists\ \alpha_1, \dots, \alpha_k \in \mathbb{C}$ & $n_1, \dots, n_k \in \mathbb{N}$, as well as $a \in \mathbb{C}$, $\neq 0$ such that: $n_1 + \dots + n_k = n$

$$Q_1(z) = a(z-\alpha_1)^{n_1} \dots (z-\alpha_k)^{n_k}.$$

We say that α_j is a root of $Q_1(z)$ with multiplicity n_j.

So, we can do this for our $P_1(z)$ & $Q_1(z)$. If $P_1(z)$ & $Q_1(z)$ have some common zeros (roots), we can simplify them. So we can write:

$$\frac{P_1(z)}{Q_1(z)} = \frac{P_2(z)}{Q_2(z)}$$

where $\deg P_2 < \deg Q_2$ & P_2, Q_2 have no common roots.

Step 3: Assume:

$$Q_2(z) = a(z-\alpha_1)^{n_1} \dots (z-\alpha_k)^{n_k}$$

Then $\exists$ constants $c_{i,j}$ $(i = 1, \dots, k,\ j = 1, \dots, n_i)$ so that:

$$\frac{P_2(z)}{Q_2(z)} = \sum_{i,j} \frac{c_{i,j}}{(z-\alpha_i)^j}$$

How to find $c_{i,j}$?

$$f(z) = \frac{z^3}{z^2-2z+1} = f_1(z) + \frac{P_1(z)}{Q_1(z)}$$

$f_1(z) = z+2$

$P_1(z) = 3z-2 = 3\left(z-\frac{2}{3}\right)$

$Q_1(z) = z^2-2z+1 = (z-1)^2$

$P_1(z)$ & $Q_1(z)$ have no common roots, so we can write $\frac{P_1(z)}{Q_1(z)} = \frac{c_{1,1}}{z-1} + \frac{c_{1,2}}{(z-1)^2}$

How to find $c_{1,1}$ & $c_{1,2}$?

What happens if we multiply with $(z-1)^2$?

$$(z-1)^2 \cdot \frac{P_1(z)}{Q_1(z)} = c_{1,1}(z-1) + c_{1,2}$$

So formally, if $H(z) = \frac{(z-1)^2 P_1(z)}{Q_1(z)}$ then:

$H(1) = c_{1,2}$

$H'(z) = c_{1,1} \Rightarrow H'(1) = c_{1,1}$.

$H(z) = \frac{(z-1)^2 P_1(z)}{Q_1(z)}$ ← the root of $Q_1(z)$

$= \frac{(z-1)^2}{(z-1)^2} \frac{(3z-2)}{} = 3z-2$

$c_{1,2} = H(1) = 1$, $H'(z) = 3$

$c_{1,1} = H'(z)|_{z=1} = 3$

$$\frac{P_1(z)}{Q_1(z)} = \frac{3}{z-1} + \frac{1}{(z-1)^2}$$

Generalise this calculation: if $Q_2(z) = (z-\alpha)^n$

If $\frac{P_1(z)}{Q_1(z)} = \frac{P_1(z)}{(z-\alpha)^n} = \frac{c_{1,1}}{z-\alpha} + \frac{c_{1,2}}{(z-\alpha)^2} + \dots + \frac{c_{1,n}}{(z-\alpha)^n}$

$c_{1,1}, \dots, c_{1,n} \in \mathbb{C}$

Multiplying both sides with $(z-\alpha)^n$:

$$(z-\alpha)^n \frac{P_1(z)}{Q_1(z)} = \sum_{i=1}^{n} \left(\frac{c_{1,1}}{z-\alpha} + \frac{c_{1,2}}{(z-\alpha)^2} + \dots + \frac{c_{1,n}}{(z-\alpha)^n} \right)$$

$$\Rightarrow P_1(z) = c_{1,1}(z-\alpha)^{n-1} + c_{1,2}(z-\alpha)^{n-2} + \dots + c_{1,n-1}(z-\alpha) + c_{1,n}.$$

So: $P_1(\alpha) = c_{1,n}$

& $P_1'(z) = (n-1)c_{1,1}(z-\alpha)^{n-2} + c_{1,2}(n-2)(z-\alpha)^{n-3} + \dots + c_{1,n-2} \cdot 2(z-\alpha) + c_{1,n-1}$

$c_{1,n-1} = P_1'(\alpha)$

Taking derivatives again:

$P_1''(z) = c_{1,1}(n-1)(n-2)(z-\alpha)^{n-3} + \dots + c_{1,n-3} \cdot 3 \cdot 2 (z-\alpha) + c_{1,n-2} \cdot 2$

$\Rightarrow P_1''(\alpha) = 2 c_{1,n-2}$

Continuing to take derivatives gives us all $c_{1,1}, c_{1,2}, \dots, c_{1,n}$.

What if: $Q_2(z) = (z-\alpha_1)^{n_1}(z-\alpha_2)^{n_2}$?

Then:

$$\frac{P_2(z)}{Q_2(z)} = \left(\frac{c_{1,1}}{z-\alpha_1} + \dots + \frac{c_{1,n_1}}{(z-\alpha_1)^{n_1}} \right) + \left(\frac{c_{2,1}}{z-\alpha_2} + \dots + \frac{c_{2,n_2}}{(z-\alpha_2)^{n_2}} \right)$$

If we want to find $c_{1,j}$, we multiply both sides with $(z-\alpha_1)^{n_1}$:

$$H_1(z) = \frac{(z-\alpha_1)^{n_1}}{} \frac{P_2(z)}{Q_2(z)}$$

Need to factor out $(z-\alpha_1)^{n_1}$ in $Q_2(z)$

$\Rightarrow c_{1,n_1} = H_1(\alpha_1)$

$c_{1,n_1-1} = H_1'(\alpha_1)$

$c_{1,n_1-2} = \frac{H_1''(\alpha_1)}{2!}$ ...

another method: L'Hospital rule

Similarly, for to find out C_{2j}, we multiply both sides with $(z-\alpha_2)^{n_2}$

$$H_2(z) = (z-\alpha_2)^{n_2} \cdot \frac{P_2(z)}{Q_2(z)}$$

$$\Rightarrow C_{2,n_2} = H_2(\alpha_2)$$
$$C_{2,n_2-1} = H_2'(\alpha_2)$$
$$C_{2,n_2-2} = \frac{H_2''(\alpha_2)}{2!} \dots$$

If $Q_2(z)$ has more roots, e.g. $Q_2(z) = (z-\alpha_1)^{n_1}(z-\alpha_2)^{n_2}(z-\alpha_3)^{n_3}$

then we can do similarly.

$$\frac{P_2(z)}{Q_2(z)} = \left(\frac{C_{11}}{z-\alpha_1} + \dots + \frac{C_{1,n_1}}{(z-\alpha_1)^{n_1}}\right) + \left(\frac{C_{21}}{z-\alpha_2} + \dots + \frac{C_{2,n_2}}{(z-\alpha_2)^{n_2}}\right) + \left(\frac{C_{31}}{z-\alpha_3} + \dots + \frac{C_{3,n_3}}{(z-\alpha_3)^{n_3}}\right)$$

Can use softwares to find roots of a polynomial $Q_2(z)$ to calculate $H_1(z) = (z-\alpha_1)^{n_1} \cdot \frac{P_2(z)}{Q_2(z)}$ & compute $H_1'(z), H_1''(z), \dots$ (Wolfram Alpha)

Step 4:

Need to find power series for:

+ A polynomial $f_1(z)$: itself! (Don't need to do anything).

+ $\frac{C}{(z-\alpha)^n}$ → Don't need to care about C, so just need $\frac{1}{(z-\alpha)^n}$.

$$\frac{1}{(z-\alpha)^n} = \frac{(-1)^n}{(\alpha-z)^n} = \frac{(-1)^n}{\alpha^n} \cdot \frac{1}{(1-\frac{z}{\alpha})^n}$$

If $n=1$ (geometric series):

$$\frac{1}{1-\frac{z}{\alpha}} = 1 + \frac{z}{\alpha} + \left(\frac{z}{\alpha}\right)^2 + \left(\frac{z}{\alpha}\right)^3 + \dots$$

If $n=2$? Take derivative: HW

$n=3$: Take more derivative HW.

Ex: $f_1(z) + \frac{P_1(z)}{Q_1(z)} = z+2 + \frac{3}{z-1} + \frac{1}{(z-1)^2}$

Power series for $z+2$ is itself.

Power series for $\frac{3}{z-1}$: Multiply 3 with

Power series for $\frac{1}{z-1}$:

$$\frac{1}{z-1} = -\frac{1}{1-z} = -\left(1+z+z^2+z^3+\dots\right)$$

Power series for $\frac{1}{(z-1)^2}$:

$$\frac{1}{z-1} = -\left(1+z+z^2+z^3+\dots\right)$$

Take derivative:

LHS: $-\frac{1}{(z-1)^2}$

RHS $= -\left(0\cdot 1 + 2z + 3z^2 + \dots\right)$

So $\frac{1}{(z-1)^2} = 1 + 2z + 3z^2 + 4z^3 + \dots$

Summary: $\frac{z^3}{z^2-2z+1} = (z+2)$

$f(z) = -3\left(1 + z + z^2 + z^3 + \dots\right)$

$+ \left(1 + 2z + 3z^2 + 4z^3 + \dots\right)$

$= 0 + 0z + 0z^2 + z^3 + 2z^4 + \dots$