20 March 2023
MAT 2250

Maze problem:

$G = (V, E)$ connected graph.

We want to start from a vertex u_0, go to all the edges, once for each direction, then come back to u_0.



Example: a postman starts from 1 (which is the post office) & wants to deliver mails to all the houses 2,3,4,... & comes back to the post office after delivering all.

Maze-solving algorithm on Wikipedia (which is actually for a related but different problem, & which you see more in newspapers.)

Other related problems: Eulerian path, Hamilton path, Travelling salesman problem (TSP).

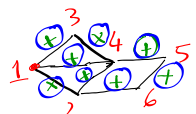
Ancient method: We will follow these 2 rules:

(1) No edge will be traversed more than once in the same direction. (Obviously)

(2) If $u_0 \to \dots \to u_p = u \to v$ is a path already constructed, then we mark the edge $u \to v$, & we only choose $v \to x$ later in W if all other edges $v \to x$ are already marked.

(In otherwords, if $\exists$ an unmarked edge $v \to x$ then choose it, otherwise choose any eligible $v \to w$.)

$u_0 = 1$



$1 \to 3$ & mark $1 \to 3$.

So now from 3, we cannot choose $3 \to 1$, because there are other unmarked edges from 3, for example $3 \to 4$.

$1 \to 3 \to 4 \to 1 \to 2 \to 4$
$\to 5 \to 6 \to 2 \to 4 \to 3 \to 1 \to 2$
$\to 6 \to 5 \to 4 \to 1$

Proposition: The algorithm with the above 2 rules solve the maze problem.

Proof:

Let W be a complete path constructed by this algorithm:

$W = u_0 \to u_1 \to \dots \to u_p$.

(Here "complete" means that we cannot extend to some edge $u_p \to u_{p+1}$.)

Observation: $u_p = u_0$.

Let $d^+_W(u)$ = the in-degree of u in W (which we think of as a graph itself)

Let $d^-_W(u)$ = the out-degree of u in W.

Claim 1: $d^+(u) = d^-(u)$ for all $u \neq u_0$.

Why?:

$\rightarrow u \rightarrow$ (everytime we come to u, then we come out of it)

Claim 2: $u_p = u_0$.

If not, then we have:

$\rightarrow u_p$ But then it means: $d^+(u) = d^-(u) + 1$

a contradiction to Claim 1.

Claim 3: For $d^+(u_0) = d^-(u_0)$ also.

$\rightarrow u_0$ (counted for $d^+(u_0)$) $\rightarrow u_0$ (counted for $d^-(u_0)$).

(If u_0 is in the middle, when we come in, then we also come out.)

Now we will

From Rule 1, each edge will only be traversed at most once in each direction.

So to show that this path W solves the maze problem, it suffices to show that:

$\forall u \in V$:

$$d^+(u) = d^-(u) = d(u)$$

(# of edges having u as 1 vertex)

$S = \{u \in V : d^+(u) = d^-(u) = d(u)\}$.

We want to show that $S = V$.

First:

Claim 4: $u_0 \in S$.

$u_0 \rightarrow \cdots \rightarrow u_0$

If $u_0 \notin S$: $d^+(u_0) = d^-(u_0) < d(u_0)$.

Then $\exists$ an edge & direction $u_0 \rightarrow x$ not in W_1, so we can extend W to

$W' = u_0 \rightarrow \cdots \rightarrow u_0 \rightarrow x$, ($W$)

which contradicts that W is complete.

Claim 5: $\forall u \in V$, then $u \in S$.

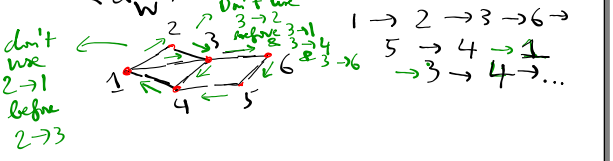
If not, let v be the first vertex $\in W$, so that $v \notin S$. So $v \neq u_0$.

$u_0 \rightarrow \cdots \rightarrow u_{\ell} = u \rightarrow u_{\ell+1} = v \rightarrow \cdots \rightarrow u_0$

So $u \in S$. Now we show that $v \rightarrow u$ is not in W_1 and hence $d^+(u) < d(u)$, so $u \notin S$.

Since we first see v in W through the edge $u \rightarrow v$, by Rule 2 we won't use the edge $v \rightarrow u$ anywhere in W before we used other edges $v \rightarrow x$.

So, because $d^-_W(v) < d(v) \Rightarrow \exists$ $v \rightarrow x_0$ so that $v \rightarrow x_0$ does not appear in W, so by Rule 2, also $v \rightarrow u$ not appear in W. This means that $d^+_W(u) < d(u) \Rightarrow u \notin S$ a contradiction. (Q.E.D.)



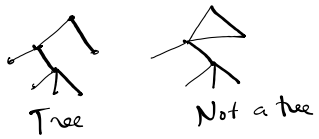
Chapter 7: Trees

Many questions in graph can reduce to tree.

§7.1 What is a tree?

Tree: A graph with no circuit. (no cycle.)

Example:

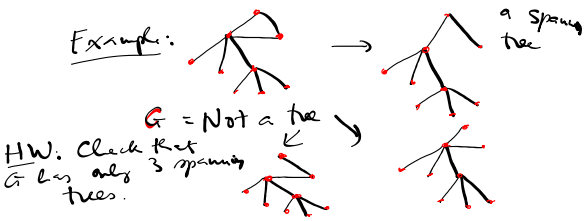


Tree Not a tree

Forest: A graph whose all components are trees is a forest.

Spanning tree: $G = (V, E)$ a connected graph which $T = (V, E')$ a subgraph of G is a tree, is called a spanning tree.

Example:



HW. Check that G has only 3 spanning trees.

Proposition: If G is a tree & H is a connected subgraph $\Rightarrow$ H is also a tree. (but with less vertices, for example).

Proof: Since G has no circuit, any subgraph also has no circuit. (Q.E.D.)

Proposition: If G is a connected graph, then it has a spanning tree.

Proof: If G has no circuit, then it is a tree, & hence is itself spanning tree. Otherwise, G has a circuit which contains an edge k.

Let $G' = (V, E \setminus \{k\})$.



Claim: G' is still connected. G' has the same vertices as G, so any

any spanning tree for G is also a spanning tree for G. then we can do induction on $|E|$.

$|E'| = |E \setminus \{k\}| = |E| - 1$.

G = ~~Connected~~ (Q.E.D.)

Theorem 7.2: The FE following are equivalent

(1) G = a tree

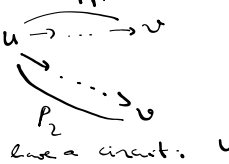
(2) If $u, v \in V \Rightarrow \exists$ one & only one path $u \to \dots \to v$ in G.

(3) G is connected & $|E| = |V| - 1$.

Proof:

(1) $\Rightarrow$ (2): G is a tree $\Rightarrow$ it is connected. So there will be at least 1 path $u \to \dots \to v$.

If $u \to \dots \to v$ is not unique,
So we have 2 p_1 paths:



So we have a circuit: $\underbrace{u \to \dots \to v}_{P_1} \underbrace{\to \dots \to u}_{P_2^{-1}}$
a contradiction.

(2) $\Rightarrow$ (3): Since we can find a path between any u & v $\Rightarrow$ G is connected.
Need only to show that $|E| = n-1$.
let $k = u \to v$ be any edge.
Claim that $G \Rightarrow G' = (V, E \setminus \{k\})$ is not connected.

(Otherwise, there is another path $u \to \dots \to v$ which is a contradiction.)

So we can write
$G' = G_1' \sqcup G_2'$
where $G_1' = (V_1, E_1)$, $G_2' = (V_2, E_2)$
& $|V_1|, |V_2| < n$.
& G_1', G_2' also satisfy that $\forall u, v$ there is one and only one path connecting.
So we can prove by induction on the number of vertices.

$\to \begin{cases} |E_1| = |V_1| - 1 \\ |E_2| = |V_2| - 1 \end{cases}$

$\Rightarrow |E| = |E_1| + |E_2| + 1 = |V_1| - 1 + |V_2| - 1 + 1 = |V| - 1.$

(3) $\to$ (1): $= (V', E')$
Let T be a spanning tree for G.
By (2) $\to$ (3): we know:
$|E'| = |V| - 1 = n - 1$
Here we assume that $|E| = n-1$
$\Rightarrow G = T \Rightarrow G$ is a tree. (Q.E.D.)

Corollary: a) If T is a tree & $d_1, \dots, d_n$ is the degree sequence, then:
$d_1 + \dots + d_n = 2|E| = 2(n-1)$.
b) If $G = (V, E)$ is a graph with t components, then any spanning forest has $|V| - t$ edges.

How to calculate how many spanning trees a connected graph $G = (V, E)$ has?

Laplacian matrix of a graph:

(Laplacian in p.d.e: $\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$, if $f = f(x_1, \dots, x_n)$)

L is an $n \times n$ matrix ($n = |V|$) whose rows & columns represent the vertices in G.

$$L[i,j] = \begin{cases} d(i) & \text{if } i = j \\ -1 & \text{if } i \text{ \& } j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

See Spanning tree on Wikipedia.

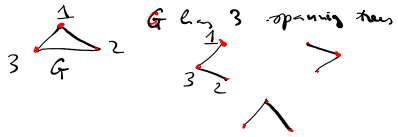
Thm (Kirchoff's theorem).

Let $\lambda_1, \cdots, \lambda_{n-1}$ be the non-zero eigenvalues of L. Then the number of spanning trees of G is

$$t(G) = \frac{\lambda_1 \cdots \lambda_{n-1}}{n}.$$

Proof: next time.

Example:



G has 3 spanning trees

$L[1,1] = 2$

$L[1,2] = -1$

similarly.

$$L = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Note:

$t(G)$ = the same as the determinant of any $(n-1)\times(n-1)$ minor.

An $(n-1)\times(n-1)$ minor of L is a matrix $L_{i,i}$ obtained from L by taking out the row i & the column i.

$$L_{1,1} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\Rightarrow t(G) = \det(L_{1,1}) = 2\times 2 - (-1)\times(-1) = 3.$$

Use Wolfram Alpha, we find that the eigenvalues of G are $3, 3, 0$.

Hence, again:

$$t(G) = \frac{3\times 3}{3} = 3.$$