23 February 2023
MAT 2250

Section 3.2: Solving (linear) Recurrences

The main idea is that if we have a linear recurrence

$$a_n T_n + a_{n-1} T_{n-1} + \cdots + a_{n-k} T_{n-k} = 0 \quad \forall n,$$

Then if we consider the generating function:

$$\hat{T}(z) = T_0 + T_1 z + T_2 z^2 + T_3 z^3 + \cdots$$

Then if $n \geq k$, we can replace

$$T_n = -\left(a_{n-1} T_{n-1} + a_{n-2} T_{n-2} + \cdots + a_{n-k} T_{n-k} \right) / a_n$$

$$\hat{T}(z) = T_0 + T_1 z + T_2 z^2 + \cdots + T_{k-1} z^{k-1} - \sum_{n=k}^{\infty} \left(a_{n-1} T_{n-1} + a_{n-2} T_{n-2} + \cdots + a_{n-k} T_{n-k} \right) z^n / a_n$$

Look for example this summand:

$$- \sum_{n=k}^{\infty} \frac{a_{n-1}}{a_n} T_{n-1} z^n$$

$$= -z \sum_{n=k}^{\infty} \frac{a_{n-1}}{a_n} T_{n-1} z^{n-1}$$

related to $\hat{T}(z)$

If this is like n^a, then we can relate this to the derivation or integration of $\hat{T}(z)$. Then we can solve for $\hat{T}(z)$.

Example:

$$T_n = 6T_{n-1} - 9T_{n-2} \quad \forall n \geq 2$$
$$T_0 = 1, \; T_1 = 2$$

$\hat{T}(z)$ = generating function for $\{T_n\}_{n=k}$

$$= T_0 + T_1 z + T_2 z^2 + T_3 z^3 + \cdots$$
$$= T_0 + T_1 z + (6T_1 - 9T_0) z^2 + (6T_2 - 9T_1) z^3 + \cdots$$
$$= T_0 + T_1 z + \hat{S}_1(z) + \hat{S}_2(z)$$

where

$\hat{S}_1(z) = 6T_1z^2 + 6T_2z^3 + \cdots$

$= 6z(T_1z + T_2z^2 + \cdots)$

$= 6z(\hat{T}(z) - T_0)$

$\hat{S}_2(z) = -9T_0z^2 - 9T_1z^3 - \cdots$

$= -9z^2(T_0 + T_1z + T_2z^2 + \cdots)$

$= -9z^2\hat{T}(z)$.

So, we have a functional equation for $\hat{T}(z)$:

$\hat{T}(z) = T_0 + T_1z + 6z(\hat{T}(z) - T_0) - 9z^2\hat{T}(z)$

Move terms with $\hat{T}(z)$ to the LHS:

$\hat{T}(z)(1 - 6z + 9z^2) = 1 - 4z$

$\hat{T}(z) = \frac{1-4z}{1-6z+9z^2} = \frac{P(z)}{Q(z)}$

is a rational function.

First $P(z)$ & $Q(z)$ have no common roots so we don't need to simplify.

Find the roots of $Q(z)$ (the denominator):

$\therefore 1 - 6z + 9z^2 = 0 \Rightarrow (1-3z)^2 = 0 \Rightarrow$

$z = \frac{1}{3}$ with multiplicity 2.

So by the general theory for to find power series representation of a rational function:

$\frac{1-4z}{1-6z+9z^2} = \frac{c_1}{z-\frac{1}{3}} + \frac{c_2}{(z-\frac{1}{3})^2}$

for constants c_1 & c_2.

$= \frac{\hat{c}_1}{3z-1} + \frac{\hat{c}_2}{(3z-1)^2}$

Remind about how to find $\hat{c}_1$ & $\hat{c}_2$:

We multiply by $(3z-1)^2$ on both sides:

$\frac{1-4z}{1-6z+9z^2} = \frac{\hat{c}_1}{3z-1} + \frac{\hat{c}_2}{(3z-1)^2}$

$\Rightarrow (3z-1)^2\left[\frac{1-4z}{1-6z+9z^2}\right] = \hat{c}_1(3z-1) + \hat{c}_2$

$\Rightarrow 1 - 4z = \hat{c}_1(3z-1) + \hat{c}_2$

Many ways to find $\hat{c}_1$ & $\hat{c}_2$ from this. The easiest way is to do like we discussed before:

To find $\hat{c}_2$: evaluate both sides at the root $z = \frac{1}{3}$:

$z = \frac{1}{3}$:

$1 - 4(\frac{1}{3}) = \hat{c}_1(3\cdot\frac{1}{3} - 1) + \hat{c}_2$ (the $\hat{c}_1$ term is crossed out: $= 0$)

$\hat{c}_2 = -\frac{1}{3}$

To find $\hat{c}_1$: first take derivative (w.r.t. z) both sides, and then evaluate at the root $z = \frac{1}{3}$:

$(1-4z)' = (\hat{c}_1(3z-1) + \hat{c}_2)'$

$-4 = 3\hat{c}_1 + 0$

$\Rightarrow -4/3 = \hat{c}_1$.

$\Rightarrow \frac{1-4z}{1-6z+9z^2} = -\frac{4}{3}\cdot\frac{1}{3z-1} - \frac{1}{3}\frac{1}{(3z-1)^2}$

$-\frac{4}{3}\cdot\frac{1}{3z-1} = \frac{4}{3}\cdot\frac{1}{1-3z} = \frac{4}{3}\left(1 + 3z + (3z)^2 + (3z)^3 + \dots\right)$ (general term 3^n)

$-\frac{1}{3}\cdot\frac{1}{(1-3z)^2}$

Remember: $\frac{1}{(1-3z)^2} = \frac{1}{3}\left(\frac{1}{1-3z}\right)'$

$\frac{1}{1-3z} = 1 + 3z + 9z^2 + (3z)^3 + \dots$

$\Rightarrow \left(\frac{1}{1-3z}\right)' = +3z + 9\cdot 2z + 3^3\cdot 3z^2 + 3^4\cdot 4\cdot z^3 + \dots$

$\frac{1}{3}\cdot\frac{3}{(1-3z)^2}$ general term $3^n \cdot n$ ($(z^n)' = n\cdot z^{n-1}$ → we see the n here because)

Adding everything, then the general expression for T_n is:

$T_n = a\cdot 3^n + b\cdot n 3^n$ for some a, b constants.

Similarly if we see something like

$\frac{c}{(1-qz)^3}$ then there will be a term of the form $\alpha\, a^n\cdot n^2 + \beta\, a^n\cdot n + \gamma\cdot a^n$.

Why?

$\frac{1}{(1-qz)^a} = 1 + qz + (qz)^2 + (qz)^3 + \dots$

$a\frac{1}{(1-qz)^2} = \left(\frac{1}{1-qz}\right)' = q + q^2\cdot 2z + q^3\cdot 3z^2 + q^4\cdot 4z^3 + \dots$

$2a\frac{1}{(1-qz)^3} = \left[\frac{1}{(1-qz)^2}\right]'$

So ⇒ we see something like:

$$a^n \cdot n(n-1) = a^n \cdot n^2 + a^n(-n)$$

Theorem 3.1: Let $q_1, \dots, q_k$ be constants. Let T_n satisfy: (homogeneous)

$$T_n = q_1 T_{n-1} + \dots + q_k T_{n-k} \quad \forall n \geq k$$

If $\hat{T}(z)$ = the generating function for the sequence $\{T_n\} = T_0 + T_1 z + T_2 z^2 + \dots$ then:

$$\hat{T}(z) = \frac{P(z)}{Q(z)}$$

for $Q(z) = 1 - q_1 z - q_2 z^2 - \dots - q_k z^k$

& if $Q(z) = -q_k (z - r_1)^{n_1} \cdots (z - r_d)^{n_d}$ (Assume $r_1, \dots, r_d \neq 0$)

where $n_1 + \dots + n_d = k$,

then $\exists$ constants α_{ij} $\left(\begin{matrix} i = 1, \dots, d \\ j = 0, \dots, n_i - 1 \end{matrix}\right)$

such that

$$T_n = \sum_{i=1}^{d} \sum_{j=0}^{n_i - 1} \alpha_{ij} \left(\frac{1}{r_i}\right)^n \cdot n^j$$

Proof: Like what we did with how to represent a rational function by power series. □

What if we don't have constant coefficients? (We may need to use differential equations, instead of functional equations)

Example: Derangement numbers D_n (see pages 44 & 45 in the book)

$$D_n = n D_{n-1} + (-1)^n, \quad n \geq 1$$

$$D_0 = 1$$

n → non-constant; $(-1)^n$ → non-homogeneous

Let $\hat{D}(z)$ = the generating function for $\{D_n\} = D_0 + D_1 z + D_2 z^2 + D_3 z^3 + \dots$

$$\hat{D}(z) = D_0 + (1 \cdot D_0 + (-1)^1) z + (2 \cdot D_1 + (-1)^2) z^2 + (3 \cdot D_2 + (-1)^3) z^3 + \dots$$

$$= D_0 + \hat{S}_1(z) + \hat{S}_2(z)$$

$$\hat{S}_1(z) = 1 \cdot D_0 z + 2 D_1 z^2 + 3 D_2 z^3 + 4 D_3 z^4$$

What is the relation to $\hat{D}(z)$?

$$\hat{D}(z) = D_0 + D_1 z + D_2 z^2 + \dots \quad \text{v.s.}$$

$$\Rightarrow \hat{D}'(z) = D_1 + 2 D_2 z + 3 D_3 z^2 + \dots$$

The terms in $\hat{S}_1(z)$ are $(1+n)D_n z^{n+1} \geq z+2$

The terms in $(\hat{D}(z))'$ are: $n D_n z^{n-1}$

So:

$$\hat{S}_1(z) = D_0 + z^2\left[(1+1)D_1 + (2+1)D_2 z + (3+1)D_3 z^2 + \dots\right]$$

$$= D_0 + z^2\left[\hat{D}(z)' + \frac{1}{z}\left(D_1 z + D_2 z^2 + \dots\right)\right]$$

$$= D_0 + z^2\hat{D}(z)' + z\left(\hat{D}(z) - D_0\right)$$

$$\hat{S}_2(z) = (-1)z + (-1)^2 z^2 + (-1)^3 z^3 + \dots$$

$$= \left[1 + (-z) + (-z)^2 + (-z)^3 + \dots\right] - 1$$

$$= -1 + \frac{1}{1+z}$$

$$\Rightarrow \hat{D}(z) = 2D_0 + z^2\hat{D}(z)' + z\left(\hat{D}(z) - D_0\right) - 1 + \frac{1}{1+z}$$

involving $\hat{D}(z)$, $\hat{D}(z)'$ & known functions & is linear in $\hat{D}(z)$ & $\hat{D}'(z)$.

This is a linear differential equation in $\hat{D}(z)$ of order 1 (i.e. involving up to the 1-st derivative) & non-homogeneous.

We won't study about Differential equations in this course (maybe Wolfram Alpha can solve for you).

Need $\hat{D}(0)$, $\hat{D}'(0)$.

$\hat{D}(0) = D_0$, $\hat{D}'(0) = D_1$