23 March 2023
MAT 2250

Theorem (Kirchoff): Let $\lambda_1, \dots, \lambda_{n-1}$ be the non-zero eigenvalues of the Laplacian matrix of a connected graph G. Then, the number $t(G)$ of spanning trees of G is:

$$t(G) = \frac{\lambda_1 \cdots \lambda_{n-1}}{n} = \det(L_{i,i})$$

$L_{i,i}$ is a minor of L.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Proof: First we show for example that $\det(L_{1,1}) = t(G)$.

Construct the oriented incidence matrix E ($n \times m$) as follows:

$$E[i,k] = \begin{cases} 0, & \text{if } i \text{ \& } k \text{ are not adjacent} \\ 1, & \text{if } i \text{ \& } j \text{ are adjacent \& } k = ij,\ i < j \\ -1, & \text{if } i \text{ \& } j \text{ are adjacent \& } k = ij,\ i > j \end{cases}$$

(vertex i, edge k)

Then Claim: $L = EE^T$.

$$L[i,j] = \begin{cases} d(i) & \text{if } i = j \\ -1 & \text{if } i \text{ \& } j \text{ are adjacent} \\ 0 & \text{if } i \neq j \text{ \& not adjacent} \end{cases}$$

$$EE^T[i,j] = \sum_{k \in E} E[i,k]\, E^T[k,j] = \sum_{k \in E} E[i,k] E[j,k]$$

(edge $k \in E$)

If $i = j$: $EE^T[i,i] = \sum_{k \in \text{edges}} E[i,k]E[i,k]$, where $E[i,k]E[i,k] = 1$ if $i \in k$, 0 otherwise

$$= \sum_{\substack{k \in \text{edges} \\ i \in k}} 1 = d(i).$$

If i & j not incidence $\Rightarrow \forall k$: $E[i,k] = 0$ or $E[j,k] = 0 \Rightarrow EE^T[i,j] = 0$

If i & j incidence: $\exists$ at most one k so that in that case $k = ij$, $E[i,k]E[j,k] = 1$, & none of $E[i,k]$, $E[j,k]$ is -1, the other is -1.

F = $(n-1) \times n$ matrix obtained from E by deleting the 1st row.

Then $FF^T = L_{1,1}$. Proof:

$EE^T = L \Rightarrow EE^T[i,j]$ $(i,j > 1)$

$= E[i,\cdot] \cdot E[j,\cdot]$

(row i of E in F; row j of E in F)

(Rows in F are the same as in E, except the 1st row of E.).

For each subset S of $n-1$ elements of $\{1, \dots, m\}$, let F_S be the $(n-1) \times (n-1)$ matrix obtained from the columns indexed by S in F.

Example, if $n = 4$

$F =$ 3×4 matrix $\begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{pmatrix}$

S = subset of 3 elements in $\{1,2,3,4\}$.

Example $S = \{1,2,4\} \Rightarrow F_S$

Then by Cauchi-Binet formula for computing determinant of a product of 2 matrices (like $L_{1,1} = FF^T$)

$\det(L_{1,1}) = \det(FF^T) = \sum_S \det(F_S)^2$.

Each S gives us possibly $n-1$ edges of the graph G.

$|V| = 5, |E| = 4$



$E =$ 5×4 matrix

	I	II	III	IV
1	1	0	0	0
2	-1	1	0	1
3	0	-1	1	0
4	0	0	-1	0
5	0	0	0	-1

$F =$

	I	II	III	IV
2	-1	1	0	1
3	0	-1	1	0
4	0	0	-1	0

$F_{2,3,4,5}$: $\begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Remember:

$\det(M) = \sum_{\sigma} \text{sign}(\sigma)\, m_{1,\sigma(1)} m_{2,\sigma(2)} \cdots m_{n,\sigma(n)}$

σ: a permutation of $\{1,\dots,n\}$

expand $\det(F_S)$, there is at most 1 σ for which $(-1)^{\text{ord}(\sigma)} m_{1,\sigma(1)} \cdots m_{n,\sigma(n)} \neq 0$, that gives us the spanning tree.

σ: permutation $\Rightarrow \sigma = [\sigma_1][\sigma_2]\cdots[\sigma_k]$ → transposition $[x \leftrightarrow y]$

$\text{sign}(id) = 1$

$\text{sign}((x \leftrightarrow y)) = -1$

$\text{sign}(\sigma) = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases}$

Now we show that L has an eigenvalue 0, & if $\lambda_1, \dots, \lambda_{n-1}$ are non-zero eigenvalues of L then:

$t(G) = \frac{\lambda_1 \cdots \lambda_{n-1}}{n}$.

Proof. $L = \begin{pmatrix} d(1) & 0 & -1 & \cdots \\ & d(2) & & \end{pmatrix}$

The sum of all entries in any row is 0.

Meaning $L \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 0.

If $\lambda_n = 0$, $\lambda_1, \dots, \lambda_{n-1}$ are all the eigenvalues of L, then the characteristic polynomial $p(t)$ of L can be computed in 2 ways:

(*) $p(t) = (t-\lambda_1)\dots(t-\lambda_{n-1})(t-\lambda_n)$ ($\lambda_n = 0$)
$= t(t-\lambda_1)\dots(t-\lambda_{n-1})$.

(**) $p(t) = \det(tId - L)$

From (*) $\Rightarrow$ the coefficient of t in $p(t)$ is $(-1)^{n-1}\lambda_1\dots\lambda_{n-1}$.

From (**) $\Rightarrow$ the coefficient of t in $p(t)$ is $(-1)^{n-1}(\det(L_{1,1}) + \det(L_{2,2}) + \dots + \det(L_{n,n}))$

But we know that

$\det(L_{1,1}) = \dots = \det(L_{n,n}) = t(G)$

$\Rightarrow (-1)^{n-1}\lambda_1\dots\lambda_{n-1} = (-1)^{n-1} n \cdot t(G)$

$\Rightarrow \dfrac{\lambda_1\dots\lambda_{n-1}}{n} = t(G)$.

Example $n = 3$:

$p(t) = \det(tId - A) = \det\begin{pmatrix} t-a_{11} & -a_{12} & -a_{13} \\ -a_{21} & t-a_{22} & -a_{23} \\ -a_{31} & -a_{32} & t-a_{33} \end{pmatrix}$

What is the coefficient of t in $p(t)$? (Q. E.D)

§7.2. Search algorithms in trees

We know how many spanning trees a connected graph $G = (V, E)$ has.

The question is how to find a spanning tree.

Two methods:

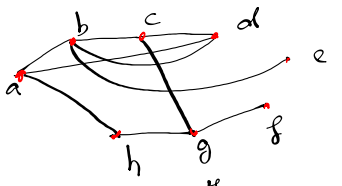
Breadth-first search:

Step 0: Choose arbitrarily a vertex u_0.

Step 1: Then add all neighbors of u_0 in (together with edges)

Step 2: Then add all neighbors of those in Step 1, not appeared before together with edges.

Keep going until we cannot do any more.



Step 0: $c = 1$ Breadth-first search

Step 1: $b = 2$, $d = 3$, $g = 4$

$a = 5$, $e = 6$, $f = 7$, $h = 8$

Depth-search first:

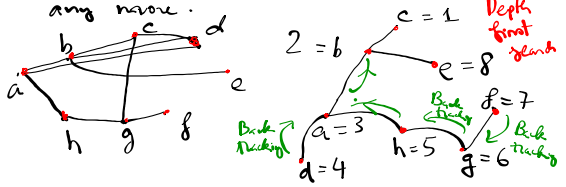
Step 0: Choose arbitrarily u_0

Step 1: If u_i is already chosen, then choose arbitrarily u_{i+1} adjacent to u_i which does not appear before

If u_i has no adjacent point not appearing before $\Rightarrow$ we step back (Backtracking) to ~~u_i~~ & one which is connected to u_i in the path we constructed, & try to find a new adjacent vertex to u_{i-1}.

If not possible for u_{i-1} then we step one step more & so on.

Repeat Step 1 until we cannot do any more.



Theorem: If G is connected, then both Breadth-first search & Depth-first search find spanning trees for G.

Proof: Quite obvious. Check it has no circuit & connected.

Look at the pictures & explain! □

E.g.: For breadth-first search, if v not in the list yet, we can find a path from $u_0 \to v$, & that path needs to be considered at a vertex which still has some adjacent points not yet in the list. (Q.E.D.)

§ 7.3 Minimal spanning tree.

This is more general & useful than the ones considered in § 7.2, in that we allow weighted trees.

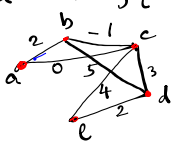
Q: ~~Find a~~ Given a ~~weighted graph~~ $G = (V, E, w)$: $w: E \to \mathbb{R}$ is a weight

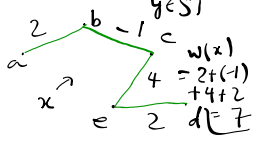
(w gives each edge a real number, called weight.)

Let ST = the set of all spanning tree of G.

If $x \in ST$ $w(x) = \sum_{e \in x} w(x)$

Find $x \in ST$ such that $w(x) = \min_{y \in ST} w(y)$.





Definition (Matroid): is a pair

$$\mathcal{M} = (S, \mathcal{U})$$

where

S = a finite set

$\mathcal{U} \subseteq 2^S$ ~~subset of the~~ collection of subsets of S, satisfying some ~~real~~ conditions:

① $\phi \in \mathcal{U}$

② $A \in \mathcal{U}$, $B \subseteq A \Rightarrow B \in \mathcal{U}$.

③ $A, B \in \mathcal{U}$ & $|B| = |A| + 1$ $\Rightarrow \exists\, k \in B$ s.t. $A \cup \{k\} \in \mathcal{U}$.

A matroid is an abstraction & generalization of a matrix.

Examples:

From set theory:

S = finite set.

$\mathcal{U} = 2^S$

called a free matroid. Examples

From linear algebra: (Basically ① & ② are the same.)

① Vector matroid:

V = a vector space over a field F.

S a finite subset of V.

$\mathcal{U} = \{ Y \subseteq S : \text{vectors in } Y \text{ are linearly independent} \}$

$V = \mathbb{R}^3$ (v_1, v_2, v_3, v_4)

$S = \{(1,2,3), (0,1,0), (0,0,1), (1,1,1)\} \subseteq V$

$\mathcal{U} = \{ \emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\} \}$

② Column matroid:

Look at a matrix A

& S = the set of columns of A.

& $\mathcal{U}$ = subsets of S whose corresponding columns are linearly independent.

Exmpls ② & ① are basically the same.

For example, Example ① (vector matroid):

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 \end{pmatrix}$$

$S = \{1, 2, 3, 4\}$.

Representable matroid: Any matroid which is equivalent to a vector matroid.

There is also matroid from number theory where "independent" means "algebraically independent".

The theorem 7.13 which we will learn applies to every matroid! Ex: to number theory matroid.