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MAT 2250

§ Summary

+ Counting
+ Probability
  expectation $\leftarrow E[X] = \sum_x p(x)\, x$
  variation $\leftarrow V[X] = E[X^2] - (E[X])^2$.
+ Generating function
  If $\{a_n\}_{n=0}^{\infty}$ is a sequence, then the generating function is $F(t) = \sum_{n=0}^{\infty} a_n t^n$.

The idea is that to find $F(t)$ is maybe easier than to find a_n's directly. Knowing $F(t)$ we can go back & find an individually.

Linear recurrent Partial fraction decomposition

Eg: $a_0 = 2,\ a_1 = 1,\ a_2 = 0$

$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3},\ n \geq 3$

$a_3 = 3 \times 0 - 3 \times 1 + 2 = -1$

$a_4 = 3 \times (-1) - 3 \times 0 + 1 = -2$

...

How to find $\{a_n\}$?

First, try to find the generating function:

$F(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots$

$F(t) = 2 + t + 0t^2$

$(3a_2 - 3a_1 + a_0)t^3 + (3a_3 - 3a_2 + a_1)t^4$

$+ (3a_4 - 3a_3 + a_2)t^5 + \dots$

$= 2 + t$

$+ (3a_2 t^3 + 3a_3 t^4 + 3a_4 t^5 + \dots)$

$+ (-3a_1 t^3 - 3a_2 t^4 - 3a_3 t^5 - \dots)$

$+ (a_0 t^3 + a_1 t^4 + a_2 t^5 + \dots)$

Green are $= 3t(a_2 t^2 + a_3 t^3 + a_4 t^4 + \dots)$

$= 3t(F(t) - 2 - t)$

Red are $= -3t^2(a_1 t + a_2 t^2 + \dots)$

$= -3t^2(F(t) - 2)$

Blue are $= t^3(a_0 + a_1 t + a_2 t^2 + \dots)$

$= t^3 F(t)$

$F(t) = 2 + t$

$+ 3t(F(t) - 2 - t)$

$+ -3t^2(F(t) - 2)$

$+ t^3 F(t)$

Move $F(t)$ to LHS:

$F(t)(1 - 3t + 3t^2 - t^3)$

$= 2 + t - 6t - t^2 + 6t^2$

$= 2 - 5t - 5t^2$

Solve for $F(t)$:

$F(t) = \dfrac{2 - 5t - 5t^2}{1 - 3t + 3t^2 - t^3}$

Next step: Find partial fraction decomposition for $F(t)$.

First step: Find the roots of the denominator

$1 - 3t + 3t^2 - t^3 = (1-t)^3$.

3 roots are the same = 1

$\Rightarrow$ We can write:

$$F(t) = \frac{2 - 5t - 5t^2}{1 - 3t + 3t^2 - t^3} = \frac{a}{1-t} + \frac{b}{(1-t)^2} + \frac{c}{(1-t)^3}$$

where a, b, c are constants.

Two common ways to find a, b, c.

First method: RHS $= \dfrac{a(1-t)^2 + b(1-t) + c}{(1-t)^3}$

$\Rightarrow 2 - 5t - 5t^2 = a(1-t)^2 + b(1-t) + c$

$= a(t^2 - 2t + 1) + b(-t + 1) + c$

$= t^2 a + t(-2a - b) + (a + b + c)$

Two polynomials are the same iff the coefficients (of powers of t) are the same.

$a = -5$ (coefficient of t^2)

$-2a - b = -5$ (coefficient of t)

$a + b + c = 2$ (constant coefficient)

$\Rightarrow$ Solve for a, b, c.

$a = -5, \; b = 15, \; c = -8$.

Another way:

$2 - 5t - 5t^2 = a(1-t)^2 + b(1-t) + c$

if we put $t = 1$:

LHS $= -8$, RHS $= c$

$\Rightarrow 2 - 5t - 5t^2 = a(1-t)^2 + b(1-t) - 8$

Take derivative w.r.t. t:

$-5 - 10t = -2a(1-t) - b$

if we put $t = 1$

LHS $= -15$, RHS $= -b$

$\Rightarrow b = 15$

$\Rightarrow -5 - 10t = -2a(1-t) - 15$

Take derivative w.r.t. t:

LHS $= -10$, RHS $= 2a$

$\Rightarrow a = -5$

$$F(t) = \frac{2 - 5t - 5t^2}{1 - 3t + 3t^2 - t^3} = \frac{-5}{1-t} + \frac{15}{(1-t)^2} - \frac{8}{(1-t)^3}$$

Next step: Find power series of RHS.

Can do this for each term.

$$\frac{-5}{1-t} = (-5) \times \frac{1}{1-t} \overset{\text{power series}}{=} (-5)(1 + t + t^2 + t^3 + \dots)$$

$$\frac{15}{(1-t)^2} = 15 \times \frac{1}{(1-t)^2} \quad \left(\sum_{n=0}^{\infty} t^n \right)$$

How to find power series for $\dfrac{1}{(1-t)^2}$:

Way 1 (may be time consuming):

$$\frac{1}{(1-t)^2} = \frac{1}{1-t} \times \frac{1}{1-t} =$$

$$(1 + t + t^2 + \dots) \times (1 + t + t^2 + \dots)$$

$$= 1 + 2t + 3t^2 + 4t^3 + \dots$$

$$= \sum_{n=0}^{\infty} (n+1) t^n$$

prove by induction

Another way:

$$\left(\frac{1}{1-t}\right)' = \frac{1}{(1-t)^2}$$

$$\left(1+t+t^2+\cdots\right)' = \frac{1}{(1-t)^2} \Leftarrow$$

$$1+2t+3t^2+\cdots$$

Apply the 2nd way to find p.s. for $\frac{1}{(1-t)^3}$:

$$\left((1-t)^{-2}\right)' = \left(\frac{1}{(1-t)^2}\right)' = \frac{2}{(1-t)^3}$$

$$(1+2t+3t^2+\cdots)' = 2+6t+12t^2+\cdots = \frac{2}{(1-t)^3} \Leftarrow$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)t^n$$

in summary:

$$F(t) = -5\sum_{n=0}^{\infty}t^n + 15\sum_{n=0}^{\infty}(n+1)t^n - \frac{8}{2}\sum_{n=0}^{\infty}(n+1)(n+2)t^n$$

$$= \sum_{n=0}^{\infty}\left(-5+15(n+1)-4(n+1)(n+2)\right)t^n$$

$$\sum_{n=0}^{\infty}a_n t^n \Rightarrow a_n = -5+15(n+1)-4(n+1)(n+2) \quad \forall n$$

Check with $n=0,1,2,3$ to make sure

$$a_n = O(n^2)$$

$$= \Omega(n^2)$$

If the roots are not 1, 1, 1, but say 1, 2, 3, then:

$$F(t) = \frac{a}{1-t} + \frac{b}{2-t} + \frac{c}{3-t}$$

If the roots are 1, 2, 2:

$$F(t) = \frac{a}{1-t} + \frac{b}{2-t} + \frac{c}{(2-t)^2}$$

How to write $\frac{1}{2-t}$ in p.s.?

$$\frac{1}{1-s} = 1+s+s^2+s^3+\cdots$$

$$\frac{1}{2-s} = \frac{1}{2}\cdot\frac{1}{1-\frac{s}{2}} = \frac{1}{2}\left(1+\frac{s}{2}+\left(\frac{s}{2}\right)^2+\cdots\right)$$

+ Graphs:

Connected or not?

Directed or undirected?

Weighted or not?

Multigraph or not?

Degree of a vertex = # of edges having that vertex

Multigraph

Loop

Adjacency matrix: $A = n \times n$ matrix (n = # of vertices)

$$A_{ij} = \begin{cases} 1 & \text{if } \exists \text{ an edge between } i \text{ \& } j \\ 0 & \text{otherwise} \end{cases}$$

$A_{[i,j]}$ = the number of ~~edges~~ paths of length 1 (= edge) between i & j.

$A^2_{[i,j]}$ = # of paths of length 2 between vertices i & j.

$A^3_{[i,j]}$ = # ... length 3 ... i & j.

Incidence matrix B = $n \times q$ matrix

n = # of vertices

q = # of edges

$B_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ belongs to edge } j \\ 0 & \text{otherwise.} \end{cases}$

$\deg(i) = \sum_{j=1}^{q} B_{ij}$

Path: P = a list of vertices $v_1, \dots, v_k$

where $v_i \neq v_j$, $i \neq j$

length of P = # of edges $v_i v_{i+1}$ is an edge for all $i = 1, \dots, k-1$

Circuit: P = a list of vertices $v_1, \dots, v_k, v_{k+1}$:

$v_1, \dots, v_k$ is a path

$v_{k+1} = v_1$

$v_k v_{k+1}$ is a path

path ... circuit

Trees: = a graph + connected + no circuit

Between any two vertices i & j in a tree there is a unique path connecting i & j.

Forest = a union of trees

Rooted tree: a tree with a special vertex called a root r.

~~Eg i belo~~

The height of a vertex i is the length of the unique path from root r to i.

If ij belongs to the unique path from root r to j $\Rightarrow$ i is a predecessor of j & j is a successor of i. If moreover, ij is ~~Eg~~ an edge, then i is an immediate predecessor (parent) of j, & j is an immediate successor (child) of i. Vertices having no successor ~~is~~ are leaves.

(n, q) tree: a rooted tree with n leaves, & each vertex has $\leq q$ immediate successor.

Any not complete (n,q) tree can be completed by adding some leaves.

The black & blue one is a complete $(12,4)$ tree.

$(12,4)$ tree

* the black tree is not a complete $(12,4)$ tree

Breadth first & Depth first search

Find spanning tree of a graph:

(Breadthfirst) Start from one vertex r.

Step 1: Go to all edges having r:



Step 2: For all children of r, do again, but make sure not if a vertex is already in, then don't take it again.



To find minimal spanning tree => use Matroid (Basis with minimal weight for matroids).

Matroid is also helpful in Algebra, Linear Optimization.

Finn. no: find airline tickets from A to B.

Dijkstra algorithm to find the shortest path in a graph.

Matchings: A set of edges in a graph, no two have a common vertex.

Flows: Directed weighted graph

+ with a capacity
+ with 2 special vertices
  s = the source
  t = the sink (the terminal)

Need to remember:
+ Admissible flow
+ Value of an admissible flow

Important for questions related to network flows: the notion of increasing path (augmenting path). Need to know, how, from an augmenting path from s to t, can improve a given admissible.

Euler circuit & open Euler circuit:

⇓ (a permutation)

a sequence of q edges: (q = # of edges of the graph)

$k_1, \dots, k_q$

where $k_i \neq k_j \ \forall i \neq j$

k_i & k_{i+1} have common vertex $i = 1, \dots, q-1$

k_q & k_1 have common vertex

(if k_q & k_1 have no common vertex => open Euler circuit).

Thm: G (connected, multigraph) has an Euler circuit ⟺ Every vertex has even degree. G has an open Euler circuit iff ∃ exactly 2 vertices of odd degree. □

Need to know how to construct (copy) Euler circuit (Lecture notes, Book) Important: remember to avoid cut edge (= bridge) as much as possible.

Hamilton circuit: A circuit which contains each vertex exactly once.

Important: if the weighted graph is not complete, then add more edges (with large enough weights) to complete it. Of course, then the graph may not satisfy the triangle inequality.

Need to know: Construct a Spanning tree + Complete the spanning tree (for example: double every edge) to make sure it has an Euler circuit.

Find an Euler circuit of the above Completed doubling tree.

From an Euler circuit, delete some edges to get a Hamilton circuit.

(Read lecture notes, examples).

Also need to know how to calculate the weight of a Hamilton circuit = $\sum$ of the weights of the edges in the circuit.

(n, q) - tree:

Important theorems:

Kraft's inequality: The necessary & sufficient conditions for $L_1, \dots, L_n$ to be the lengths of leaves of an (n,q)-tree. □

Need to know construct an (n,q) tree satisfying Kraft's inequality.

→ Start from the root

→ Construct leaves of ~~tree~~ length 1. ($L_i = 1$)

→ ... length 2 ($L_i = 2$)

Huffman's algorithm: Construct an (n,q)-tree T whose leaves have pre-assigned probabilities $p_1, \dots, p_n$ such that

$$\overline{L}(T) = \sum_{i=1}^{n} p_i L(x_i) \to \text{minimum.}$$

Need to know how to do. Step 1: Start from the leaves, gradually ~~move~~ step 1, 1 expand the root to an (n,q)-tree

Shanna's theorem.

$$\underbrace{-\sum_{i=1}^{n} p_i \log p_i}_{\text{entropy}} \le \min_{T:(n,q)\text{-tree with pre-assigned probabilities at the leaves } p_1,\dots,p_n} \overline{L}_q(T) \le 1 + \underbrace{\left(-\sum_{i=1}^{n} p_i \log p_i\right)}_{\text{entropy}}$$

Sorting: → Insertion Sort; → Merge sort

Binary search trees: Organise data into a binary rooted tree ($=$ if ... then ...) $(n,2)$-tree.

Thm. If T is an (n, q)-tree

then:

$L(T) \geq \lceil \log_q n \rceil$. (...)