30 March 2023
MAT 2250

§7.4. The shortest path in a graph

Question. $G = (V, E)$ connected, weighted. u & $v \in V$

Find a path P from u to v with smallest weight. (If the weight = 1 for all edges, so we find the shortest distance.)

Algorithm (Dijkstra algorithm): Fix $u \in G$. Will construct a spanning tree T for G, with the property that the path in T from u to any v in G is a minimal path.

Step 0: $u_0 = u$, $V_0 = \{u_0\} \subseteq V$,

$E_0 = \phi \subseteq E$, $L(u_0) = 0$

$L = \{L(u_0)\}$ $L(v)$ = distance from u_0 to v

set of edges in the tree.

Step i+1. Assume that we already constructed u_i, $V_i = \{u_0, \dots, u_i\}$, $E_i = \{k_1, \dots, k_i\}$, $L = \{L(u_0), L(u_1), \dots, L(u_i)\}$, where $L(u_j)$ = minimal distance from u_0 to u_j $\forall$ $j = 1, \dots, i$.

We will now construct $k_{i+1}, V_{i+1}, E_{i+1}, \dots$

Define $A_i = \{k \in E :$ k has one vertex in V_i & 1 vertex in $V \setminus V_i\}$

Choose a $k_{i+1} = \underset{k \in A_i}{\operatorname{argmin}} [L(v) + w(k)]$

v_i k w $V \setminus V_i$

If $k_{i+1} = v_{i+1} w_{i+1}$, with $v_{i+1} \in V_i$ and $w_{i+1} \in V \setminus V_i$

$\Rightarrow$ we update:

$u_{i+1} = w_{i+1}$

$V_{i+1} = V_i \cup \{u_{i+1}\}$

$E_{i+1} = E_i \cup \{k_{i+1}\}$

$L(u_{i+1}) = L(v_{i+1}) + w(k_{i+1})$.

Repeat until we $\forall$ we exhausted all vertices of G.

This is a greedy algorithm.

Thm 7.15: Let $G = (V, E)$ be a connected graph, with non positive weights.

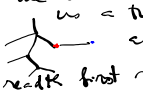
Fix $u \in V$. Then Dijkstra's algorithm produces a tree spanning tree with the property that $\forall$ $v \in V$:

$L(v) = d(u, v)$.

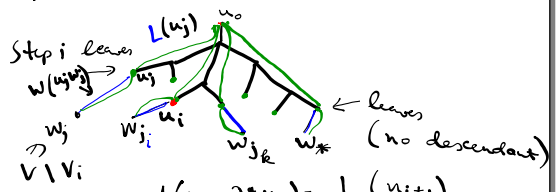
Q: What is wrong if there is an edge whose weight is negative?

Proof:

First, we show that this algorithm produces a spanning tree. Because at each step we add in an edge with one vertex inside the tree already constructed & one vertex outside it. So each step will give us a tree. And since G is connected we can cover all vertices (like Depth First Search + Breadth first search combined, we will see later.)



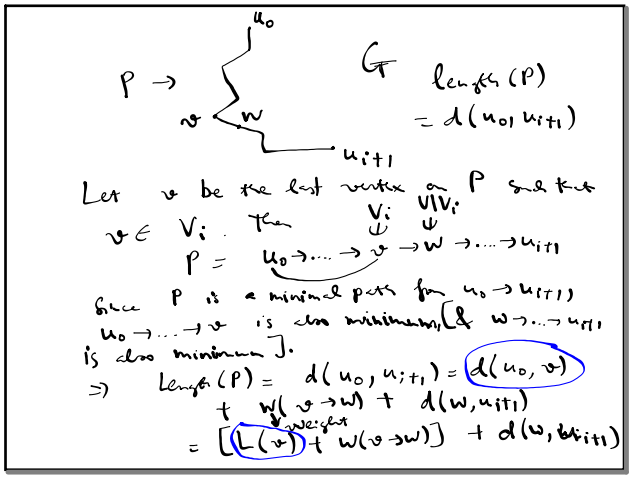
Now we show that it is minimum.



$V \setminus V_i$

Claim: $d(u_0, u_{i+1}) = L(u_{i+1})$.

Why? Just take a path in G, which connects u_0 & u_{i+1} & such that length$(P) = d(u_0, u_{i+1})$.



Let v be the last vertex on P such that $v \in V_i$. Then $v \in V_i$, $w \notin V_i$:

$P = u_0 \to \cdots \to v \to w \to \cdots \to u_{i+1}$

Since P is a minimal path from $u_0 \to u_{i+1}$, $u_0 \to \cdots \to v$ is also minimum [& $w \to \cdots \to u_{i+1}$ is also minimum].

$\Rightarrow$ length$(P) = d(u_0, u_{i+1}) = d(u_0, v)$ $+ w(v \to w) + d(w, u_{i+1})$

$= [L(v) + w(v \to w)] + d(w, u_{i+1})$ (weight)

We have:

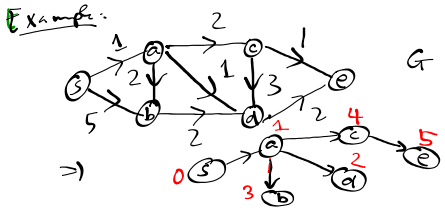
$L(v) + w(v \to w) \geq L(u_{i+1})$

$d(w, u_{i+1}) \geq 0$

$\Rightarrow d(u_0, u_{i+1}) \geq L(u_{i+1})$

Obviously, also $L(u_{i+1}) \geq d(u_0, u_{i+1})$. □

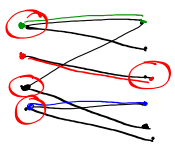
Example:



Chapter 8: Matching & Networks

§ 8.1 Matchings in bipartite graphs

Definition. $G = (V = S+T, E)$ is a bipartite graph. Then a matching M is a subset of E such that if $e_1, e_2 \in M$, $e_1 \neq e_2$, then e_1 & e_2 have no common vertex.



The above ones (green, red, blue) is a matching.

Definition (matching number):

$m(G) = \max_{M:\text{ a matching}} |M|$

Definition: (Maximum matching):

$|M| = m(G)$

Definition (neighbours): $A \subseteq S$

$\Rightarrow N(A) = \{ v \in T : \exists\, u \in A$ such that $uv \in E \}$.

Thm 8.3:

(1) $m(G) \leq |S|$

(2) $m(G) = |S|$ iff $\forall\, A \subseteq S$: $|A| \leq |N(A)|$.

Example: Look at the previous graph.

Proof:

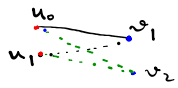
(1) is obvious. Each edge e in M must contain 1 & only 1 vertex in $S \Rightarrow$ $|M| \leq |S|$.

(2) ($\Leftarrow\!\!\Rightarrow$) If $\exists\, A \subseteq S$ such that $|A| > |N(A)|$, then like (1) since each edge e in M contains 1 & (only one) vertex from $N(A) \Rightarrow$ the edges in M cannot cover all the vertices in $A \Rightarrow$ $|M| < |S|$.

($\Leftarrow$) Assume $|A| \leq |N(A)|$ $\forall\, A \subseteq S$.

Assume that M is a matching with $|M| < |S|$. We will modify M to a matching M' with $|M'| = |M| + 1$.

Let $u_0 \in S$, $u_0 \notin M$ (i.e. $\nexists$ $e \in M$ so that u_0 is a vertex of e).



1 $\#\{\{u_0\}\} \leq |N\{u_0\}|$

$\Rightarrow \exists\, v_1 \in T$ connecting to u_0

If $v_1 \notin M \Rightarrow$ add $u_0 v_1 \in M \Rightarrow$ we let $M' = M \cup \{u_0 v_1\}$

2 $\#\{\{u_0, u_1\}\} \leq |N(\{u_0, u_1\})|$ $\Leftarrow$ If $v_1 \in M \Rightarrow \exists\, u_1 \in S$ so that $u_1 v_1 \in M$

If $v_2 \notin M$ then we ~~cancel~~ stop.

If $v_2 \in M$: $u_0 \notin M \Rightarrow u_0 v_2 \notin M$.

$u_1 v_2 \notin M$ because $u_1, v_1 \in M$, & M is a matching. $\Rightarrow \exists\, u_2 \in S$: $u_2 v_2 \in M$.

$M \not\ni u_0 \text{ —— } v_1 \in M$

$M \ni u_1 \text{ —— } v_2 \in M$

$M \ni u_2 \text{ —— } v_i \in M$ Again, since $\#\{\{u_0, u_1, u_2\}\} \leq |N(\{u_0, u_1, u_2\})|$

$M \ni u_i \text{ —— } v_{i-1} \in M \Rightarrow \exists\, v_3$ which connects to u_0, u_1 or u_2

$M \ni u_{n-1}$ $v_n \notin M$ then we keep going. We stop at some $v_N \notin M$,

So: we keep $u_{n-1} v_{n-1}, \ldots, u_i v_i, v_i$ (v_N connects to one of the $u_0, \ldots, u_{N-1}$.

Add $u_N v_i$ but delete $u_i v_i$

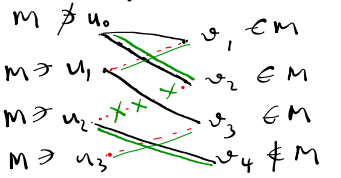
v_i connects to one of the $u_0, \ldots, u_{i-1}$, say u_i.

Now, do the same with u_0 replaced by v_i, & keep going.

Still we get the same number of edges, & still a matching.

But at the end, we have 1 more edge, which is $u_0 v_h$ (Because $u_0 \notin M$). ↓ □

For example:



(The red edges are in M).

We add $u_2 v_4$, delete $u_2 v_2$.

Add $v_2 u_0$, don't need to delete anymore ($u_0 \notin M$)

More generally:

Theorem 8.4:

$$m(G) = |S| - \max_{A \subseteq S} \{|A| - |N(A)|, 0\}$$

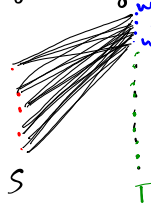
Proof:

If $|A| \le |N(A)| \ \forall A \subseteq S \Rightarrow \max \{|A| - |N(A)|, 0\} = 0$ & by Theorem 8.3 we have a matching M with $|M| = |S|$.

So only need to consider the case $|A| > |N(A)|$ for some $A \subseteq S$.

Define $\delta = \max_{A \subseteq S} |A| - |N(A)| > 0$

The trick is to reduce to the case of Theorem 8.3 by adding more points to T.



Add δ more points to T

Add edges from these new points to all vertices in S.

(Basically make these new points to be neighbors to every vertex in S.

$\Rightarrow$ get a new graph $G' = (V \cup T', E + \text{new edges})$ where $T' = T \cup \{w_1, \dots, w_\delta\}$.

If $A \subseteq S$:

$$|A| = |A| - |N_G(A)| + |N_G(A)| \le \delta + |N_{G'}(A)|$$

$$\le |N_{G'}(A)|$$

By Theorem 8.3

$\Rightarrow$ a matching M' of G' such that $|M'| = |S|$.

To go from M' to a matching M on G, we only need to delete the edges which contain one among $\{w_1, \dots, w_\delta\}$.

Maximum number of such edges is δ.

$$\Rightarrow |M| \ge |M'| - \delta = |S| - \delta$$

$$\Rightarrow m(G) \ge |S| - \delta.$$

The other inequality $m(G) \le |S| - \delta$ is obvious (like how we prove for the case $|A| > |N(A)|$ in Thm 8.3.)

Definition (Vertex cover)

$G = (V = S + T, E)$ bipartite.

$D \subseteq V$ is a vertex cover if $\forall e \in E$: e containing at least one vertex in D.

Thm 8.5 (~~Minimax theorem~~ (John Nash Game theory) theorem, equilibrium)

$$m(G) = \max_{M:\text{matching}} |M| = \min_{D:\text{vertex cover}} |D|$$