MAT 2250
4 May 2023

Hamilton circuit: Find a circuit in a graph G which visits each vertex once.

Travelling Salesman problem: Find a Hamilton circuit with minimal weight.
(For each edge, we have a non-negative weight.)

Rmk: Hamilton was not the first one to consider this problem.

Thm: Determine whether a graph has a Hamilton circuit/path is NP-complete.
Proof: Look for references. (Q.E.D.)

Informally, this means that if we can solve Hamilton circuit problem in $O(P(n))$, where $P(n)$ is some function of the size, & K is another NP-problem, then we can solve K in $O(P(n)Q(n))$ steps, where $Q(n)$ is a polynomial in n.

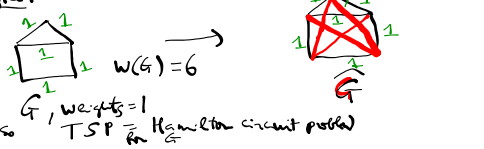
Start with some simple algorithms:
$G = (V, E)$, $|V| = n$

Obvious algorithm: Look at all possibilities of n-edges $(e_{i_1}, \dots, e_{i_n})$ where e_{i_j} & $e_{i_{j+1}}$ are coincident & e_{i_1} & e_{i_n} are also coincident. Then find the one with minimal weight.

Complexity: If $|E| = N$, then we have $N(N-1)\cdots(N-n+1)$ choices.
Too much.

Reduction to complete graphs:
$G = (V, E)$ $|V| = n$
Now if $x, y \in V$ & $x \to y$ is not in E, we add $x \to y$ & the weight $w(x \to y)$ is big enough. For example: choose $w(x \to y) = w(G) + 1$
This way we have a complete graph $\hat{G}$.
$= (V, \hat{E})$

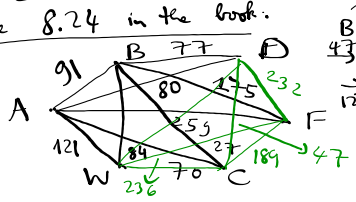
Example:



(so TSP for Hamilton circuit problem)

Nearest neighbor algorithm (NN): For complete graph
Start with any vertex, say V_{i_1}.
Now assume that we already find vertices $\{V_{i_1}, \dots, V_{i_m}\}$.
Choose V in the remaining vertices so that $V V_{i_m}$ has minimal weight. Add $V = V_{i_{m+1}}$.
Continue until we form $m = n$.

Example 8.24 in the book:



Double nearest neighbourhood algorithm:

Like before but now look for minimum of distances to both V_{i_1}, V_{i_m} if we already have $\{V_{i_1}, \dots, V_{i_m}\}$. If VV_{i_m} is minimum $\to \{V_{i_1}, \dots, V_{i_m}, V\}$. If VV_{i_1} is minimal $\Rightarrow \{V, V_{i_1}, \dots, V_{i_m}\}$.

Rmk. If the weights of edges are distinct, then start from any V_{i_1}, the NN & DNN give a unique construction.

Example (previous page)

Step 0: B $\to$
Step 1: B $\to$ C, Step 2: B $\to$ C $\to$ D,
Step 3: B $\to$ C $\to$ D $\to$ W,
Step 4: A $\to$ B $\to$ C $\to$ D $\to$ W (Different from NN)
Step 5: A $\to$ B $\to$ C $\to$ D $\to$ W $\to$ F $\to$ A

There is also an variation Cheap Insertion. (not discussed here).

There are many examples where NN & DNN give the worst route.

there are some better algorithms.

Minimal spanning tree heuristic (MST):

The main idea is: If $V_{i_1} \to \dots \to V_{i_m} \to V_{i_{m+1}} = V_{i_1}$ is a Hamilton circuit $\Rightarrow$ if $V_{i_1} \to \dots \to V_{i_m}$ is a path with n vertices $\Rightarrow T$ is a tree, (spanning tree).

$w(T) \le w$ (Hamilton circuit)

Now if T_{min} is a minimal spanning tree (so may be not a path), from §7.3 then

$w(T_{min}) \le w(T) \le w$(Hamilton circuit).

1. Find a minimum spanning tree $T_{A,min}$ for G.
2. Double the edges of T_{min} $\to$ get $T_{min,D}$.

add one more edge for each edge in T_{min}, with the same weight.

Now every vertex has even degree.

$T_{min,D}$ has $2(n-1)$ edges, $n = |V|$.

3. Let $E_{min,D}$ be an Eulerian circuit.

It may not be yet a Hamilton circuit because it may visit one vertex more than once.

We can get a Hamilton circuit by the following: Keep a vertex when it first appears in $E_{min,D}$ & delete it if it appears later.

Example: Look at the first half of page 176

Then $E_{min,D} = C \to B \to F \to \cancel{B} \to \cancel{C} \to D \to W \to \cancel{D} \to \cancel{C} \to A \to C$

$H_{min,D} = C \to B \to F \to D \to W \to A \to C$ (last arc)

Q. Where do we use that G is complete?

Christofides heuristic = MST + Minimal matching for vertices with odd degree in T_{min}:

We don't need to have $T_{min,D}$ to get an Euler circuit. We only need to make sure that every vertices have even degree. So only need to care about the set of vertices of odd degree in T_{min} & add some more edges, to make them have even degree.

1. Find a minimal spanning tree T_{min} (§7.3)

2. U = the set of vertices having odd degree in T_{min}. Also complete, with $|U|$ vertices. Find a path $G_U = (U, \forall E_U) \subseteq G$ (i.e.: vertices are in U, & edges connecting 2 vertices in U). Find a minimal weight matching M_{min}

3. The graph $T_{min} + M_{min}$ has that every vertex has even degree.
So we can find a Euler circuit $E_{min, CH}$. Then as in the MST heuristic, we get a Hamilton circuit $H_{min, CH}$.

Example: The second half of page 176.

T_{min} = A → C → D → W; C → B → F

A, C, W, F have odd degrees.
Now find a maximal matching of A, C, W, F in G with minimal weight. In this case, the matching is $M_{min} = \{A \to C,\ W \to F\}$
(Explain: Look at the table in Example 8.24 on page 174)
Match1: AC, WF → weight = 70 + 236 = 306 ← minimum
Match2: AW, CF → weight = 121 + 189 = 310
Match3: AF, CW → weight = 259 + 55 = 314
So ...

$T_{min} + M_{min}$ = A → C → D → W → B → F

$E_{min, CH}$ = B → C → A → ~~X~~ → D → W → F → B

$H_{min, CH}$ = B → C → A → D → W → F → B. ← last one

Rmk: All of these algorithms require that G is complete. All of them are polynomial.
But how is the weight imposed to minimal weight? We need an extra condition:
Triangle inequality condition: If $x \to y \to z$ is a path in G, then $w(x \to z) \le w(x \to y) + w(y \to z)$.

Rmk: Triangle inequality & completeness is in general not compatible. (G is complete)
Example: Look at the graph in the beginning of today's lecture.
So, the estimates in the next theorem may not be useful for non-complete graph.

Thm 8.27: G is a complete graph & the weights on G satisfies the triangle inequality. Let w_{min} to be the weight of an optimal solution to the TSP. Then:
a) $w(H_{MST}) \le 2 w_{min}$
b) $w(H_{CH}) \le 1.5 w_{min}$ ← World record

Proof:

a) We know

$w(T_{min}) \le w(\text{optimal Hamilton}) = w_{Hmin}$

$T_{min,D}$ = double of T_{min}

$\Rightarrow w(T_{min,D}) = 2\, w(T_{min})$.

$H_{min,D} \subseteq E_{min,D}$

By triangle inequality:

$w(H_{min,D}) \le w(E_{min,D}) = w(T_{min,D}) = 2\, w(T_{min}) \le 2\, w_{Hmin}$

(Explain with previous example)

b) Similar:

Let H = optimal Hamilton circuit

$v_1, v_2, \dots, v_n, v_{n+1} = v_1$

Let's say the vertices in T_{min} with odd degree are $v_{i_1}, \dots, v_{i_{2m}}$ $(i_1 < i_2 < \dots < i_{2m})$

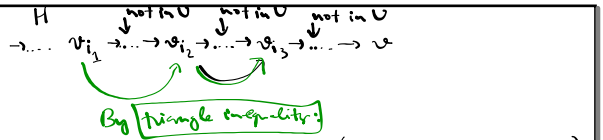
Consider 2 matchings in G of $v_{i_1}, \dots, v_{i_{2m}}$

$M_1 = \{ v_{i_1}v_{i_2}, \dots, v_{i_{2m-1}}v_{i_{2m}} \}$

$M_2 = \{ v_{i_2}v_{i_3}, v_{i_4}v_{i_5}, \dots, v_{i_{2m}}v_{i_1} \}$

$w(M_{min}) \le w(M_1)$ & $w(M_2)$

↓ minimal weight



By triangle inequality:

$w(v_{i_1}v_{i_2}) \le w(v_{i_1} \to \dots \to v_{i_2} \text{ in } H) \le w(H) = w_{Hmin}$

$\Rightarrow w(M_1) + w(M_2) \le w(H) = w_{Hmin}$

$(M_1 + M_2 \subseteq H)$ $\Rightarrow w(M_{min}) \le \frac{1}{2} w_{Hmin}$

$\Rightarrow w(E_{min,CH}) \le w(T_{min}) + w(M_{min})$ (triangle inequality) $\le w_{Hmin} + \frac{1}{2} w_{Hmin} = \frac{3}{2} w_{Hmin}$

$(E_{min,CH} \subseteq T_{min} + M_{min})$

(Q.E.D.)

Summary: What triangle inequality gives us is that if ~~H is a subgraph of G~~: ~~$w(H) \le w(G)$~~.

If H is obtained from H' by deleting some vertices, then $w(H) \le w(H')$.

If we don't have triangle inequality, then what can we prove? → No one knows.

If we don't have a complete graph, then maybe the only algorithm we could use for TSP is similar to the "drawing one" (see paper "5/4 - CH is cursed") (Look at approximation for Symmetric TSP) the best algorithm we have for complete & ~~st~~ triangle inequality graph.

However, in practice usually we can get better approximation: Wikipedia says that for graphs with millions of vertices, we may get more 1.03 x W_{min} by these algorithms.