6 February 2023

MAT 2250

§ 1.6 Existence theorems

* Pigeonhole principle & extensions
* Counting # of possibilities / Showing some probability > 0

Pigeonhole principle:

The easiest version: If we have n+1 pigeons to be in n pigeonholes, then there must be 1 pigeonhole containing at least 2 pigeons.



Proof: If each pigeonhole has at most 1 pigeon, then n pigeonholes can only contain at most $n \times 1 = n$ pigeons, but we have $n+1 (> n)$ pigeons, so it is a contradiction. □

Pigeonhole in the preimage of map version:

Look at the pigeonhole principle in another way. Let's say we have:

$P = \{1, \dots, n+1\}$ the set of pigeons.

$\mathcal{H} = \{1, \dots, n\}$ the set of pigeonholes.

Now $f: P \to \mathcal{H}$

$i \mapsto j$ = the pigeonhole which contains i $= f(i)$

Then the hole j contains two pigeons i_1 & i_2 iff $i_1, i_2 \in f^{-1}(j)$.

So the pigeonhole principle says that we can find at least one $j \in \mathcal{H}$ such that $|f^{-1}(j)| \geq 2$.

We can extend this: (any finite set)

Theorem: Let $f: N \to R$ a map, $n = |N|$, $r = |R|$ & $n > r$.

Then $\exists\, a \in R$ such that $|f^{-1}(a)| \geq \left\lfloor \frac{n-1}{r} \right\rfloor + 1$. ($N$: natural numbers, R: real numbers)

$\left\lfloor \frac{n-1}{n} \right\rfloor$ = the floor function, which gives us the largest integer which is $\leq \frac{n-1}{n}$.

Proof: Similar to that for Pigeonhole principle.

If not true, then $\forall a \in R$: $|f^{-1}(a)| \leq \left\lfloor \frac{n-1}{n} \right\rfloor$, then get a contradiction. ☐

Main point is how to use it in practice.

Example: Let $a_1, \dots, a_{n^2+1}$ be a sequence of distinct real numbers. Show that there is a subsequence $\underbrace{a_{k_1}, \dots, a_{k_{n+1}}}_{n+1 \text{ elements}}$ (such that $k_1 < k_2 < \dots < k_{n+1}$) which is a monotone sequence, i.e. either $a_{k_1} < \dots < a_{k_{n+1}}$, or $a_{k_1} > a_{k_2} > \dots > a_{k_{n+1}}$.

($n = 2$, $n^2 + 1 = 5$

$\pm 1, 9, 2, -1, x$ $\quad x \neq 1, 9, 2, -1$

Want to find $n+1 = 3$ $a_{k_1}, a_{k_2}, a_{k_3}$ such that

either $a_{k_1} < a_{k_2} < a_{k_3}$

or $a_{k_1} > a_{k_2} > a_{k_3}$

if $x > 2$: $a_{k_1} = 1 < a_{k_2} = 2 < a_{k_3} = x$

if $x < 2$: $a_{k_1} = 9 > a_{k_2} = 2 > a_{k_3} = x$

Proof:

$f: A = \{a_1, \dots, a_{n^2+1}\} \to \mathbb{N}$

$a_i \longmapsto t_i = f(a_i)$ = the largest length of an monotonely increasing subsequence starting with a_i

~~Two cases~~

Let R = Image of f.

If R contains a number $\geq n+1$ then we are done.

If not then $R \subseteq [1, 2, \dots, n]$.

Apply the stronger version of the Pigeonhole principle: $\Rightarrow \exists\, t \in [1,\dots,n]$ such that

$$|f^{-1}(t)| \geq \left\lfloor \frac{(n^2+1)-1}{n} \right\rfloor + 1 = n+1.$$

($|A| = n^2+1$, $|R| = n$)

Let's choose $k_1 < \dots < k_{n+1}$ such that $a_{k_1}, \dots, a_{k_{n+1}} \in f^{-1}(t)$.

Claim: $a_{k_1} > a_{k_2} > \dots > a_{k_{n+1}}$ (reverse property of the map f.)

For example, why $a_{k_1} > a_{k_2}$?

Because, if not, then $a_{k_1} < a_{k_2}$.

Now $f(a_{k_2}) = t$ means that we can find a sequence:

$a_{k_2,1} = a_{k_1} < a_{k_2,2} < \dots < a_{k_2,t}$

But now then we have a sequence:

$a_{k_1,1} = a_{k_1} < a_{k_2,1} < \dots < a_{k_2,t}$ ($t+1$ elements)

$\Rightarrow f(a_{k_1}) \geq t+1$

So $a_{k_1} \notin f^{-1}(t)$, a contradiction. ∎

Example $\underbrace{1, 9, 2, -1, 0}_{A}$

$f: A \to \mathbb{N}$

$1 \mapsto f(1) = 2$

$9 \mapsto f(9) = 1$

$2 \mapsto f(2) = 1$

$-1 \mapsto f(-1) = 2$

$0 \mapsto f(0) = 1$

So $f(A) \subseteq [1, 2]$

So there is $t \in [1, 2]$:

$f^{-1}(t) \geq 3$

$t = 1$ & $f^{-1}(1) = 9, 2, 0$

& $9 > 2 > 0$. ∎

Now maps can be translated into colouring.

$f: \mathbb{N} \to R = \{\text{red, green, blue, } \dots\}$

R = the set of some colours, then we say we colour $\mathbb{N}$ by R

Then we can think about N like a graph.



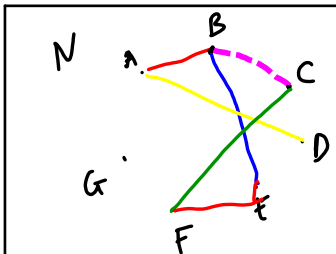
Translation of the stronger Pigeonhole Principle into colouring: Suppose we have $l_1, \dots, l_r$ & $|N| = n$, such that $n \geq l_1 + \dots + l_r - r + 1$.

Then if we colour N by r colours, then there is one colour i such that at least l_i elements of N must be have the same colour.

Proof: Similar to the above. □

The mapping version, we choose $l_1 = \dots = l_r = \left\lfloor \frac{n-1}{r} \right\rfloor \to$ colour for elements

Ramsey's theorem (for pairs of elements ↪ can be translated & even more general):



$N = \{A, B, C, D, E, F\}$

But now we don't colour the vertices. Instead, we colour pairs $\{A, B\}$, $\{A, C\}, \dots$ → can be thought of as the edges of the graph with vertices in N.

Graph theory

Ramsey's theorem: let $k, l \geq 1$. $\exists\ R(k, l)$ such that if N is a set with $|N| = n \geq R(k, l)$, then if we colour pairs in N by red or blue, then either:

$\exists\ A \subseteq N$, $|A| = k$, every pairs in A are coloured red

or

$\exists\ B \subseteq N$, $|B| = l$, every pairs in B are coloured blue.

In graph theory terminology: We have a full graph with vertices in N & we colour its edges by red or blue. If $|N| \geq R(k,l)$, then either:

$\exists$ a full subgraph with vertices in A for $|A| = k$, every edges are coloured red

or

$\exists$ . . . . . in B for $|B| = l$, every . . . . blue.

Proof: (Erdos & Szekeres)

Want Suffices to prove that

$R(k,l) \leq R(k-1,l) + R(k,l-1)$

& to show initial numbers are finite:

$R(k,2) = k < \infty$, $R(2,l) = l$.

(similar to what for the binomial numbers).

Let $|N| = R(k-1,l) + R(k,l-1)$

Choose $a \in N$.

Now

$N \setminus a = R \cup B$ (red ∪ blue)

$R = \{x \in N : \{a,x\}$ is coloured red $\}$

$B = \{y \in N : \{a,y\}$ is coloured blue $\}$ $\quad R \cap B = \emptyset$

Then $|R| + |B| = |N \setminus a|$

$= R(k,l-1) + R(k-1,l)$.

So at least $|R| \geq R(k-1,l)$ or $R(k,l-1)$

$|B| \geq R(k,l-1)$

(this is just the Pigeonhole principle in the colour version for 2 colours)

Say, $|R| \geq R(k-1,l)$

Then either we can find $B' \subseteq N$ such that $|B'| = l$ & all pairs in B' are coloured by blue (Done!)

or We can find A' such that $|A'| = l-1$ & all pairs in A' are coloured by red.

Then $A = \{a, A'\}$ has $|A| = k$, & all pairs are colored in red. □

Example: (Ramsey's theorem, Wikipedia):

$R(3,3) \leq R(2,3) + R(3,2) = 6$.

& we can find an example so that if $|V| = 5$ then ∃ no subgraph of 3 vertices every edge is colored red.

What is known about Ramsey's numbers? (arXiv: 2005.09251, Ashwin Shah)

Lower bounds (Erdos):

$R(k,k) \geq c \cdot \frac{2^{k/2}}{\sqrt{k}}$, $c > 0$ constant

Upper bounds (best now Shah):

$R(k+1, k+1) \leq e^{-c(\log k)^2} \binom{2k}{k}$

$\sim \frac{2^{2k}}{\sqrt{k}}$ ← By Stirling approximation for $\binom{2k}{k}$

There is a big gap between the upper & lower bounds.

§2 Summation

Many questions are related to taking sums.

- Integration ⇒ Riemann sum
- Riemann zeta function:

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $s \in \mathbb{C}$

$\zeta(s)$ can be first defined for $\mathrm{Re}(s) > 1$. Then we have an integral representation which can be used to define $\zeta(s)$ for other $s \in \mathbb{C}$.

Euler: $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\sum_{n=1}^{\infty} \frac{1}{n^4}$, ...

There is a so called special values of zeta function.

What is $\sum_{n=1}^{\infty} \frac{1}{n}$?

$S(1) = \infty$ Usually, this is said to be divergent.

(In this course we only consider finite sums). $\sum_{n=1}^{N} \frac{1}{n}$ we will see later.

$\sum_{n=1}^{\infty} \frac{1}{n}(-1)^n$, $S(-1) = -\frac{1}{12}$

Methods: → Direct
→ Finite difference (like integration by parts)
→ Inverse formula
→ Inclusion-Exclusion
→ Series: (Chapter 3)

§2.1 : Direct methods

Induction:
separating the last term

Summation factors (used for affine recurrence)

Induction What is

$S(n) = \sum_{k=1}^{n} k$?

$S(1) = 1$
$S(2) = 1+2 = 3$
$S(3) = 1+2+3 = 6$
$S(4) = 1+2+3+4 = 10$ ...

$S(n) \leq n^2$ (by induction)

So if we assume that $S(n)$ is a polynomial of degree 2 in n: $S(n) = an^2 + bn + c$ for some a, b, c

$1 = S(1) = a + b + c$
$3 = S(2) = 4a + 2b + c$
$6 = S(3) = 9a + 3b + c$

$\Rightarrow a = \frac{1}{2}, b = \frac{1}{2}, c = 0.$

So we "guess" that $S(n) = \frac{n^2}{2} + \frac{n}{2}$

We prove by induction:
it is true for $n = 1$:
$1 = S(1) = \frac{1^2}{2} + \frac{1}{2}$
Now assume that it is true for n.
Will show that it is also true for $n+1$:

$$S(n+1) = S(n) + n+1$$
$$= \frac{n^2}{2} + \frac{n}{2} + n+1$$

Compare with $\frac{(n+1)^2}{2} + \frac{n+1}{2}$, the same. □

Example: $k \in \mathbb{N}$

$\sum_{m=1}^{n} m^k$ is a polynomial of ~~degree~~ n in degree $k+1$.

What for $\sum_{m=1}^{n} m^2$?

Separating the last term (a bit technical)

Use mainly for geometric sums

Example:
What is

$$S(k) = \sum_{k=0}^{m} x^k \quad @ = 1 + x + \dots + x^k$$

$$S(k+1) = 1 + x + \dots + x^{k+1}$$
$$= 1 + x[1 + \dots + x^k]$$
$$S(k+1) = 1 + x\,S(k) \quad (*)$$
$$S(k+1) = 1 + x + \dots + x^{k+1}$$
$$= [1 + x + \dots + x^k] + x^{k+1}$$
$$= S(k) + x^{k+1} \quad (**)$$

So from (*) & (**):

$$S(k) + x^{k+1} = 1 + x\,S(k)$$

We can solve for $S(k)$:

$$S(k)(1-x) = 1 - x^{k+1}$$
$$S(k) = \frac{1 - x^{k+1}}{1 - x}$$