9 February 2023
Math 2025

(Back to §1.6)

Another method: Calculate the probability / the number of possibilities & show that it is positive:

Let X be a finite set
$\mathcal{F} \subseteq 2^X$: a subset of the set of all subsets of X

such that $\forall A \in \mathcal{F}$: $|A| = d$.

Say that $\mathcal{F}$ is 2-colorable if $\exists$ coloring of elements of X such that in every set $A \in \mathcal{F}$: both colors occur.
(So if all elements of A are colored by the same color then it is not acceptable)

Claim: If $|\mathcal{F}| \leq 2^{d-1} \Rightarrow$ $\mathcal{F}$ is 2-colorable.

Idea: look at all possible colorings of X by 2 colors. Call M,

Then among those, count the maximum possible, call it Max, number of coloring in which at least one $A \in \mathcal{F}$ which is colored by only 1 color (we don't like these colorings)

If $Max < M \Rightarrow$ we are done.

Let's call $n = |X|$.
$\Rightarrow$ $M = 2^n$ ways to color X by 2 colors.

let $A \in \mathcal{F} \Rightarrow |A| = d$
The number of ways to color X such that all elements of A are colored by only 1 color is:
$2 \times 2^{n-d}$.

If $|\mathcal{F}| \leq 2^{d-1} \Rightarrow \exists$ at most

$$Max < \sum_{A \in \mathcal{F}} 2 \times 2^{n-d}$$

(why $<$ but not $\leq$, because these ways are not disjoint.) $= 2 \times 2^{n-d} \times 2^{d-1} = 2^n = M$,

Ways to color X by 2 colors such that $\exists\ A \in F$ which is 1-colored.

So the number of ways to color X so that every $A \in F$ will have both colors is $\geq M - \text{Max} > 0$.

Talking about probability: $1 > \frac{M - \text{Max}}{M} > 0$. □

Comeback to Section 2.1:

Summation factor (Mostly used to find elements of linear recurrence)

Main idea:

Say we have a linear recurrence

$a_n T_n + b_n T_{n-1} - c_n = 0$ (*)

We want to make a transformation

$T_n \to S_n$ (will be specified later)

So that (*) becomes:

$\boxed{S_n = S_{n-1}} + d_n$. (***)

So now it is easy S_n from d_n:

$S_n = (S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \cdots + (S_2 - S_1) + S_1$

(telescoping series)

$= d_n + d_{n-1} + \cdots + d_2 + S_1$

How do we transform (*) to (***)?

Just make $\hat{a}_n = b_n$.

Choose a sequence s_n:

$\begin{cases} s_{n-1} a_{n-1} = s_n b_n \ \forall n. \\ s_0 = 1 \end{cases}$ (Provided $a_n, b_n \neq 0\ \forall n$)

$a_n T_n = b_n T_{n-1} + c_n$

$\Rightarrow \underbrace{s_n a_n T_n}_{S_n} = \underbrace{s_n b_n T_{n-1}}_{s_{n-1} a_{n-1} T_{n-1}} + s_n c_n$

$= s_{n-1} a_{n-1} T_{n-1} + \underbrace{s_n c_n}_{d_n}$.

Explicitly:

$s_{n-1} a_{n-1} = s_n b_n$

$\Rightarrow s_n = s_{n-1} \frac{a_{n-1}}{b_n}$

$= s_{n-2} \frac{a_{n-2}}{b_{n-1}} \cdot \frac{a_{n-1}}{b_n}$

$= \cdots$

$= \frac{a_0 \cdots a_{n-1}}{b_1 \cdots b_n}$. □

$d_n = s_n c_n = \frac{a_0 \cdots a_{n-1} c_n}{b_1 \cdots b_n}$.

$S_n = (S_n - S_{n-1}) + \cdots + (S_2 - S_1) + S_1$

$= d_n + \cdots + d_2 + S_1$

$S_1 = s_1 a_1 T_1$ ← given

$s_n a_n T_n = S_n \Rightarrow T_n = \dfrac{S_n}{s_n a_n}$

Example: Find a formula for T_n where

$T_n - 3T_{n-1} + 2 = 0$

$T_n = 3T_{n-1} - 2, \; T_1 = 2.$

(One way is through the roots of the characteristic polynomial: $t - 3$.) Wikipedia

$T_n = 3T_{n-1} - 2$

$a_n = 1 \; \forall n$

$b_n = 3 \; \forall n$

$s_n b_n = s_{n-1} a_{n-1} \quad \forall n, \; s_0 = 1$

$s_1 = \dfrac{s_0 a_0}{b_1} = \dfrac{1 \cdot 1}{3} = \dfrac{1}{3}$

$s_2 = \dfrac{s_1 a_1}{b_2} = \dfrac{1}{3} \cdot \dfrac{1}{3} = \left(\dfrac{1}{3}\right)^2$

$\vdots$

$s_n = \left(\dfrac{1}{3}\right)^n$

$c_n = -2 \; \forall n \Rightarrow d_n = s_n c_n = -2\left(\dfrac{1}{3}\right)^n$

Now if $S_n = s_n a_n T_n$ then:

$S_n = S_{n-1} + d_n$

$S_1 = s_1 a_1 T_1 = \dfrac{1}{3} \cdot 1 \cdot 2 = \dfrac{2}{3}$

$S_n = d_n + d_{n-1} + \cdots + d_2 + S_1$

$= -2\left(\dfrac{1}{3}\right)^n - 2\left(\dfrac{1}{3}\right)^{n-1} - \cdots - 2\left(\dfrac{1}{3}\right)^2 + \dfrac{2}{3}.$

Can we geometric series to write this in a more compact form.

Example: Calculate D_n, from the previous chapter. D_n = # of fixed point free permutations of $\{1, \ldots, n\}$.

Idea: Find a linear recurrence between D_n & D_{n-1}, & use the summation factor method.

Claim: $D_n = (n-1)(D_{n-1} + D_{n-2})$

$D_1 = 0$?

$D_2 = 1$? $[1 \to 2]$

For convenience, put $D_0 = 1$.

Let π be a fixed point free permutation of $\{1, \ldots, n\}$.

Then $\pi(1) \neq 1$.

$\pi(1) := i, \quad i \in \{2, \dots, n\}$

There are 2 cases:

Case 1: ~~$\pi(1) = i$~~ $\pi(i) = 1$.

$\Rightarrow \pi = [1 \to i] \circ \pi'$, ($\pi \to \pi'$ free)

π' is a fixed-point free permutation of $\{1, 2, \dots, n\} \setminus \{1, i\}$.

(there we see the summand $(n-1) D_{n-2}$)

↳ have $n-1$ choices for $i \in \{2, \dots, n\}$

Case 2: $\pi(i) \neq 1$

$\pi(i) = j$, where $j \neq 1, i$.

So $\pi = \begin{pmatrix} 1 & 2 & \dots & i & \dots & n \\ i & \pi(2) & & j & & \pi(n) \end{pmatrix}$

↓ delete, ↓ replace

$\pi' := \begin{pmatrix} 2 & \dots & 1 & & n \\ \pi(2) & \dots & j & & \pi(n) \end{pmatrix}$

π' is a fixed point free permutation of $\{1, \dots, n\} \setminus \{i\}$.

Vice versa if we have π' fixed point free permutation of $\{1, \dots, n\} \setminus \{i\}$

$\pi' = \begin{pmatrix} 2 & \dots & 1 & \dots & n \\ \pi'(2) & \dots & \pi'(j) & & \pi'(n) \end{pmatrix}$

add ↓ ↓ replace 1 by i

$\pi = \begin{pmatrix} 1 & 2 & \dots & i & \dots & n \\ i & \pi'(2) & \dots & \pi'(1) & \dots & \pi'(n) \end{pmatrix}$

Then π is a fixed point free permutation of $\{1, \dots, n\}$

& this map $\pi' \to \pi$ is the inverse of the map $\pi \to \pi'$ in previous page.

(So we see the summand $(n-1) D_{n-1}$).

So $D_n = (n-1) D_{n-2} + (n-1) D_{n-1}$

Not yet a linear recurrence between D_n & D_{n-1}.

Need to transform to a linear recurrence between of order 1.

Claim: $D_n = n D_{n-1} + (-1)^n$.

$a_n = 1$, $b_n = n$, $c_n = (-1)^n$

(Example: By using induction: check OK for $n = 1$.

Assume $D_n = n D_{n-1} + (-1)^n$.

Need to show $D_{n+1} = (n+1) D_n + (-1)^{n+1}$

Use $D_{n+1} = n D_n + n D_{n-1}$.)

Then using summation factor method:

$\Rightarrow D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

# of derangements D_n / $n!$ $= \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

$n \to \infty$ $e^{-1} = \frac{1}{2.71..} > \frac{1}{3}$

# of derangements / total # of permutations

Probability of a randomly chosen permutation of $\{1,\dots,n\}$ will have no fixed point. □

§2.2 : Calculus of finite difference

This is an analog of Calculus (differtiation, integration)

Calculus

Differentiation $\frac{df}{dx}$

Integration $\int_a^b f(x)\,dx$

Calculus of finite difference

Finite difference Δf

$\sum_{x=a}^{b} f(x)\,\Delta x$, $\Delta x = 1$

Calculus of finite difference is a discretisation of calculus. It is possible because there are some deep relations between integration & summation.

Integration = limit of summation (Riemann sums)

Summation = discretisation of integration ($dx \to \Delta x = 1$)

↓ infinitesimal Leibnitz version of Calculus

$f' = \frac{df}{dx} \Rightarrow \Delta f$

Fundamental theorem of calculus:

If $g(x) = f'(x)$, then $\int_a^b g(x)\,dx = f(b) - f(a)$.

What is the Fundamental Theorem of Calculus of Finite differences?

Need discretisation version of $f'(x)$.

Forward difference:

Δ : $f(x) \mapsto f(x+1) - f(x)$

(Think about $f'(x) = \frac{df}{dx} \Rightarrow df \hookrightarrow \Delta f$, $dx \hookrightarrow \Delta Id(x) = (x+1) - x = 1$.)

Backward difference:

∇ : $f(x) \mapsto f(x) - f(x-1)$

Fundamental Theorem of Calculus of Finite Difference:

If $g(x) = \Delta f(x)$

$\Rightarrow \sum_{x=a}^{b} g(x) = f(b+1) - f(a)$.

(a bit difference to Fundamental Theorem of Calculus)

Proof: Just the Telescoping series. □

Example: What is $1^{\underline{n}} + 2^{\underline{n}} + \dots + k^{\underline{n}}$?

$k^{\underline{n}} = k(k-1)\dots(k-n+1)$

$n=1$:

$1^{\underline{n}} + 2^{\underline{n}} + \dots + k^{\underline{n}} = 1 + 2 + \dots + k$

$n=2$:

$1^{\underline{n}} + 2^{\underline{n}} + \dots + k^{\underline{n}} = 0 + 2\cdot\underline{1} + 3\cdot 2 + 4\cdot 3 + \dots + k(k-1)$

Claim: $\Delta x^{\underline{n+1}} = (n+1)x^{\underline{n}}$.

$x^{\underline{n+1}} = x(x-1)\dots(x-n)$

$\Delta x^{\underline{n+1}} = (x+1)^{\underline{n+1}} - x^{\underline{n+1}}$

$= \boxed{(x+1)}(x)\dots(x-n+1) - x\dots(x-n+1)\boxed{(x-n)}$

$= x\dots(x-n+1)\big((x+1)-(x-n)\big)$

$= (n+1)x^{\underline{n}}$ (constant)

$\Rightarrow x^{\underline{n}} = \frac{1}{n+1}\Delta x^{\underline{n+1}}$

$= \Delta \frac{x^{\underline{n+1}}}{n+1}$

By FTOCOFD (Fundamental Theorem of Calculus of Finite Difference):

$\sum_{x=a}^{b} x^{\underline{n}} = \frac{(b+1)^{\underline{n+1}}}{n+1} - \frac{a^{\underline{n+1}}}{n+1}$.

Notice: $\Delta f(x)$ may not be easy to compute if f is not a rational function. (Rational function $= \frac{P(x)}{Q(x)}$, P, Q: polynomials)

Example: $f(x) = e^x$

$f'(x) = e^x$ (geometric series)

$\sum_{x=1}^{n} e^x$ can be easily computed Δf ? $= e^{x+1} - e^x$

$= e^x(e-1)$

Example: $f(x) = \cos(x)$

$f'(x) = -\sin(x)$

$\Delta f = \cos(x+1) - \cos(x)$

$= \cos(x)\cos(1) - \sin(x)\sin(1) - \cos(x)$

??

~~What is~~ $\sum$ ~~?~~ $\Delta \sin(x)$?

HW: Look also at $\Delta\cos(x)$, $\Delta\sin(x)$ to How can we use complete $\sum \cos(x)$?