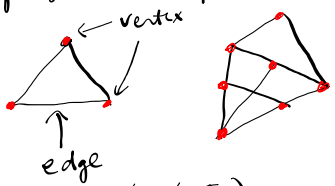
06 March 2023,
MAT 2250

§6. Graphs

§6.1: Definitions & Examples

We knew that many questions in combinatorics can be reduced to graphs. For example, Ramsey's numbers are for colouring of graphs.

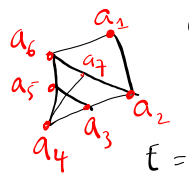
Most common: planar graphs, which we can represent on a plane.



Graph: $G = (V, E)$

V = a set of points, called vertices of the graph G

E = a subset of the set of 2-sets $\{u, v\}$ of $V \times V$.



$G = (V, E)$

$V = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$

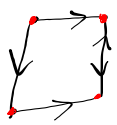
$E = \{\{a_1, a_2\}, \{a_1, a_6\}, \{a_2, a_3\}, \{a_2, a_7\}, \{a_3, a_4\}, \{a_3, a_5\}, \{a_4, a_5\}, \{a_4, a_7\}, \{a_5, a_6\}, \{a_6, a_7\}\}$

$|V| = 7$, $|E| = 10$.

undirected, no self-loop (no edge $\{u, u\}$), no weight (everything is weight 1 & each edge is counted once.)

Generalisations: (not considered in this course)

Directed graphs:

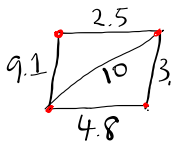


Example: 1-way streets

With self-loop: self-loop



Weighted:



Example: If you drive on a certain road, then you need to pay a certain road fee (toll).

Special graphs:

Complete graphs: E consists of all 2-subsets of V. All the complete graphs with the same number of vertices are isomorphic. (no self-loops)

K_2 :

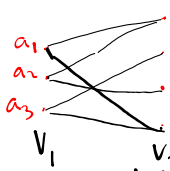
K_3 :

K_4

K_n = the complete graph with n vertices.

Bipartite graphs: $G = (V, E)$ where $V = V_1 + V_2$ (disjoint union) such that if $u_1, u_2 \in V_1 \Rightarrow u_1u_2 \notin E$, $u_1, u_2 \in V_2 \Rightarrow u_1u_2 \notin E$

In other words, an edge can only connect a point in V_1 & a point in V_2.



$V = V_1 + V_2$

$V_1 = \{a_1, a_2, a_3\}$

$V_2 = \{b_1, b_2, b_3, b_4\}$

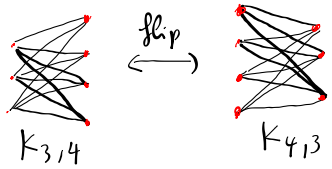
$E = \{\{a_1, b_1\}, \{a_1, b_4\}, \{a_2, b_1\}, \{a_2, b_3\}, \{a_3, b_2\}, \{a_3, b_4\}\}$.

Complete bipartite graph: $G = (V = V_1 + V_2, E)$

E consists of all pairs $\{u, v\}$, $u \in V_1$ & $v \in V_2$.

All the complete bipartite graphs with $m_1 = |V_1|$, $|V_2| = m_2$ fixed, are isomorphic, & called K_{m_1, m_2}.

Also K_{m_1, m_2} is isomorphic to K_{m_2, m_1}.



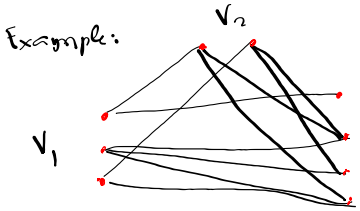
$K_{3,4}$ $K_{4,3}$

3 - partie graphs:

$G = (V = V_1 + V_2 + V_3, E)$

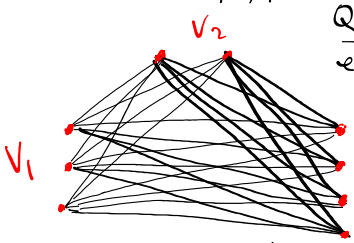
E will connect any 2 points in the same set V_i.

Example:



Complete 3 - partie graphs. K_{m_1,m_2,m_3}

Example: $K_{2,3,4}$



Q: How many edges are there in $K_{2,3,4}$?

Between V_1 & V_2: $3 \times 2 = 6$ edges

Between V_1 & V_3: $3 \times 4 = 12$ edges

Between V_2 & V_3: $2 \times 4 = 8$ edges

In total: $6 + 12 + 8 = 26$ edges.

(Complete) 4 - partie, 5 - parties ... graphs

Bipartie graphs: good for job assignment

V_1: set of employees

V_2: set of tasks need to be done.

Higher dimensional graphs: (not going in detail in this course)



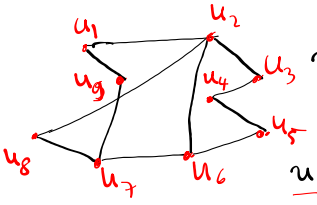
A path in a graph: $G = (V, E)$ a graph

A path in G is a sequence of vertices

$u_1, u_2, \ldots, u_k$, $u_i \in V$

$u_i u_{i+1} \in E$

$u_i \neq u_j$ for $i \neq j$



u_1, u_2: a path

u_1, u_2, u_8: a path

u_1, u_2, u_7: not a path

not an edge

$u_1, u_2, u_8, u_7, u_9, u_1$

: not a path

(u_1: appears twice)

Circuit: $u_1, \dots, u_k, u_{k+1} = u_k$

where $u_i u_{i+1} \in E$

$u_i \neq u_j$ if $i \neq j \leq k$.

Example: in the previous example:

$u_1, u_2, u_8, u_7, u_5, u_1$: a circuit.

(closed path)

But $u_1, u_2, u_8, u_7, u_5, u_1, u_2$:

not a circuit.

(2 vertices u_1, u_2 each appearing twice)

Graph isomorphisms:

$G_1 = (V_1, E_1)$

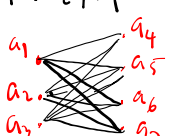
$G_2 = (V_2, E_2)$

We say that G_1 is isomorphic to G_2 iff $\exists$ bijection $f: V_1 \to V_2$ such that $\forall\ u_1, v_1 \in V_1$, then

$u_1 v_1 \in E_1 \iff f(u_1) f(u_2) \in E_2$.

(In order words, if we can renumber the vertices but we keep the edges. So all graphs $G' = (V, E)$ which are isomorphic to $G = (V, E)$ can be realised by a permutation $f: V \to V$.)

Example: K_{m_1, m_2} is isomorphic to K_{m_2, m_1}.



$K_{3,4}$



$K_{4,3}$

Permutation: $a_1 \mapsto a_5$, $a_2 \mapsto a_6$, $a_3 \mapsto a_7$

Question: Are there algorithms to decide whether two given graphs are isomorphic? What is the complexity?

Naive algorithm: $G = (V, E)$ & $G' = (V', E')$

Just consider all permutations f of V: $f: V \to V$. (all bijections $f: V \to V'$)

($\forall f$) For each of them get a representation G_f of G. Compare it with G'. If $G_f = G'$ for one $f \Rightarrow G$ is isomorphic to G'. If $\forall f$: $G_f \neq G' \Rightarrow G$ is not isomorphic to G'.

Complexity estimate: # of bijections $f: V \to V'$ is $n!$, where $n = |V| = |V'|$.

For each bijection $f: V \to V'$ we need to check that $Ef = E'$, so need to do d comparisons, where $d = |E| = |E'|$.

So the complexity $\sim n! \times d$.

(Very huge)

So this only works if n is small, e.g.: $n = 6$.

Look for graph isomorphism in Wikipedia.

What's known?

We don't know yet if it is a hard problem or not. More precisely, we don't know if it is polynomial complexity or NP-complete.

Subgraph isomorphism: $G = (V, E)$, $G' = (V', E')$.

Can we decide whether there is a subgraph $H = (V_1, E_1)$ (i.e.: $V_1 \subseteq V$, $E_1 \subseteq E$) of G so that H is isomorphic to G'.

Subgraph isomorphism problem is NP-complete.

State of the art on graph isomorphism problem:

Algorithms by Laszlo Babai (& extended by others): quasi-polynomial complexity $O\left(2^{(\log n)^3}\right)$.

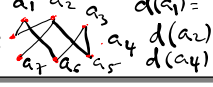
[If it were: $O\left(2^{\log n}\right)$ then it is polynomial. Because $2^{\log n} = \left(e^{\log 2}\right)^{\log n} = \left(e^{\log n}\right)^{\log 2} = n^2$.]

For planar graphs: J. Hopcroft & J. Wong (1974): efficient polynomial time solution. (Recall that usually we will use only planar graphs.)

Some relations between vertices & edges:

Neighbouring vertices: $u, v \in V$ are neighbours if $uv \in E$.

Degree of a vertex: $d(u) = \#$ of neighbours of u. $= \#$ of edges with an end point at u.



$d(a_1) = 2$

$d(a_2) = 3$

$d(a_4) = 0$

Isolated vertex:

u is isolated if $d(u) = 0$.

Incidence between a vertex & an edge:

$u \in V_1$, $k \in E$:

u & k are incident iff $u \in k$ (an end point of k)

Previous example:

$u = a_1$ & $k = a_1a_6$ are incident.

$u = a_1$ & $k = a_3a_5$ are not incident

Incidence between 2 edges:

$k_1, k_2 \in E$.

k_1 & k_2 are incident iff k_1 & k_2 have one common end point.

(Recall: an edge has 2 endpoints).

Previous example:

a_1a_8 & a_1a_6 are incident

a_1a_8 & a_2a_7 are not incident.

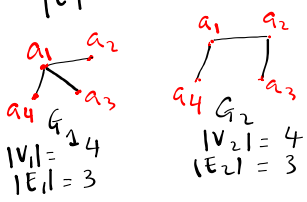
Graph invariants: A graph invariant is an object which is the same for all graphs which are isomorphic.

What to use for invariants? A good & easy way to check that 2 things are not isomorphic!

For a graph $G = (V, E)$:

$|V|$ a graph invariant,

$|E|$ a graph invariant.



But G_1 is not isomorphic to G_2.

A more sophisticated invariant:

degree sequence of a graph:

Say $G = (V, E)$

$V = \{a_1, \dots, a_n\}$

We order/arrange in such a way that

$d(a_1) \geq d(a_2) \geq \dots$

Then we call $d(a_1), d(a_2), \dots$ the degree sequence of the graph G.

For the previous example:

For G_1: $d(a_1) = 3$, $d(a_2) = 1$, $d(a_3) = 1$, $d(a_4) = 1$.

So the degree sequence for G_1 is
$d(G_1) = (3,1,1,1)$.

For G_2: $d(a_1) = 2$, $d(a_2) = 2$,
$d(a_3) = 1$, $d(a_4) = 1$.

So the degree sequence for G_2 is
$d(G_2) = (2,2,1,1)$.

$d(G_1) \neq d(G_2) \Rightarrow G_1$ is not isomorphic to G_2.

Theorem 6.1: Let $G = (V, E)$. Then $\sum_{u \in V} d(u) = 2|E|$.

An edge has 2 end points:



So this edge is counted 1 for $d(u)$ & 1 for $d(v)$.

So an edge appears 2 in the sum: $\sum_{u \in V} d(u)$.

So $\sum_{u \in V} d(u) = 2|E|$. □

Corollary: The number of vertices whose degree is an odd number is even.

Proof:

$$2|E| = \sum_{u \in V} d(u) = \left(\sum_{\substack{u \in V \\ d(u):\text{even}}} d(u)\right) + \sum_{\substack{u \in V \\ d(u):\text{odd}}} d(u)$$

$2|E|$: even number

$\sum_{u \in V} d(u)$: even

Do modulo 2 (i.e. n divide by 2 & take the remainder.

So $n \equiv 0 \pmod 2$ if n is even,
$n \equiv 1 \pmod 2$ if n is odd.)

A sum of odd numbers is even iff the number of summands is even. □