Discrete Mathematics MAT 2250
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Permutation: A permutation of $\{1,2,\dots,n\}$ is a bijective map:
$f: \{1,2,\dots,n\} \to \{1,2,\dots,n\}$

Usually, we represent f by $\{f(1), f(2),\dots,f(n)\}$

Example: $(1,2,3)$ is a permutation of the set $\{1,2,3\}$. It corresponds to the identity map.

$(1,3,2)$ is a permutation of $\{1,2,3\}$.
$1\mapsto 1$, $2\mapsto 3$, $3\mapsto 2$.

k-permutation:

k-permutation: ($k\le n$) It is an injective map:
$f: \{1,2,\dots,k\} \to \{1,\dots,n\}$.

Example: $(1,3)$ is a 2-permutation of $\{1,2,3\}$. Here $1\mapsto 1$, $2\mapsto 3$.

In words: a permutation is an ordered list of all elements in $\{1,\dots,n\}$.
A k-permutation is an ordered list of k elements in $\{1,\dots,n\}$.

Question: How many k-permutations of $\{1,2,\dots,n\}$?
(A permutation is an n-permutation of $\{1,\dots,n\}$. ?)

Product formula:

How many 1-permutations of $\{1,\dots,n\}$?
$\Rightarrow$ it corresponds to a set of 1 element of $\{1,\dots,n\}$.
$\Rightarrow$ $\exists$ n!

How many 2-permutations of $\{1,\dots,n\}$?
$\Rightarrow \#\{(a,b) : a,b \in \{1,2,\dots,n\}, a\neq b\}$

How many ways to choose a? n ways.
How many ways to choose b after that a is already chosen? b is chosen from the set $\{1,2,\dots,n\}\setminus\{a\} \Rightarrow n-1$ ways.

Product formula
$\Rightarrow$ We have $n(n-1)$ 2-permutations of $\{1,\dots,n\}$.

Example: if $n=4$ (tree diagram)



In general:
of k-permutations of $\{1,\dots,n\}$ ($k\le n$):
$= n(n-1)\dots(n-k+1) = n^{\underline{k}}$ (falling factorial)

rising factorial $n^{\overline{k}} = n(n+1)\dots(n+k-1)$

Prop: $\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$.

Why? Because $\binom{n}{k}$ = # of k-subsets of $\{1,\dots,n\}$ (i.e. # of subsets with k elements of $\{1,\dots,n\}$). (Rule of equality)

A k-subset $\Leftrightarrow$ $k!$ k-permutations

Ex: $\{1,2\}$ 2-subset $\Leftrightarrow$ $2!$ 2-permutations $(1,2)$ & $(2,1)$

$\Rightarrow$ # k-subsets = $\frac{\#\ k\text{-permutations}}{k!}$. □

$S_{n,k}$ = # of k-partitions of $\{1,\dots,n\}$
= # of ways to decompose $\{1,\dots,n\}$ into k blocks of non-empty subsets. (order not matter)

A version is called ordered k-partition:

$1+3$ is a 2-partition of 4.

If we think that $1+3 \neq 3+1$ then we have an ordered 2-partition.

Usually, the ordered ones are now easier to count than the unordered counterparts.

Prop: # of ordered k-partition of n is $\binom{n-1}{k-1}$.

Usually, the way to prove is to use induction.

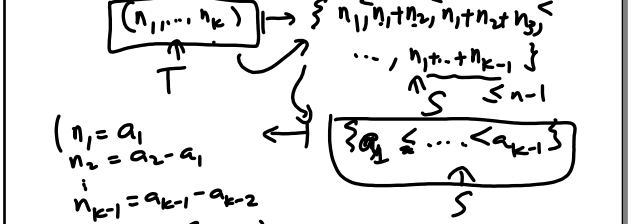
Proof: Use the Rule of equality.

$T = \{(n_1,\dots,n_k)$: a k-partition of n i.e. $n_1,\dots,n_k > 0, \in \mathbb{N}$, $n_1+\dots+n_k = n\}$

$S = \{(k-1)$-subsets of $\{1,\dots,n-1\}\}$
$= \{\{a_1,\dots,a_{k-1}\} \subseteq \{1,\dots,n-1\}\}$

$|S| = \binom{n-1}{k-1}$

So by Rule of Equality, it suffices to construct a bijection $f: T \to S$.



$(n_1 = a_1,\ n_2 = a_2 - a_1,\ \dots,\ n_{k-1} = a_{k-1} - a_{k-2},\ n_k = n - a_{k-1})$

Remind: (From MAT1140) if we have 2 maps $f: S \to T$ & $g: T \to S$ such that $f \circ g = id_S$, $g \circ f = id_T$ then f & g are bijective.

k-multiple subset: Like k-subset, but now the elements are allowed to be the same.

$X = \{1,2,3,4,5\}$

2-subset: $\{1,2,3\}$

3-multiple subset: $\{1,1,2\} = \{1,2,1\}$, $\{1,1,1\}$

Multiplicity = # of times an element appears in a multiple subset.

This happens if we have a set of objects

Example: We have 3 different nationalities Norwegian, US & UK. Want to establish a committee with 4 people. How many ways?

$\{N,N,N,N\}$, $\{N,N,N,US\}$, $\{N,N,N,UK\}$, ...

(4-multiple subset of $\{1,2,3\}$)

Prop: # of k-multiple subsets of $\{1,\dots,n\}$

$$= \frac{n^{\overline{k}}}{k!}.$$

$n^{\overline{k}} = \underbrace{n \cdots n}_{k \text{ times}}$

$n^{\overline{k}} = n(n-1)\cdots(n-k+1)$

$n^{\underline{k}} = n(n+1)\cdots(n+k-1)$

Proof: $S = \{$k-multiple subsets of $\{1,\dots,n\}\}$

$T = \{$k-subsets of $\{1,\dots,n,n+1,\dots,n+k-1\}\}$

$$|T| = \frac{n^{\overline{k}}}{k!} = \frac{(n+k-1)^{\underline{k}}}{k!}$$

An element of k-multiple subset is $\{a_1 \le a_2 \le \dots \le a_k\}$

$f: S \to T$

$\{a_1 \le \dots \le a_k\} \mapsto \{a_1 < a_2+1 < a_3+2 < \dots < a_k+k-1\}$, $a_k+k-1 \le n+k-1$

$a_1 = b_1$

$a_2 = b_2 - 1$

$\vdots$

$a_k = b_k - (k-1)$

$\leftarrow \{b_1 < b_2 < \dots < b_k\}$

If we compose, then we get identity maps. $\square$

Recall: A k-subset $\{a_1 < \dots < a_k\}$

A k-multiple subset $\{a_1 \le \dots \le a_k\}$.

Counting mappings: Say $|N| = n$,

$|R| = r$ elements

Consider maps from $N \to R$ with some special properties?

$M(N,R) = \{ f : N \to R \}$

$Inj(N,R) = \{ f : N \to R$ injective$\}$, (only possible if $n \le r$)

$Surj(N,R) = \{ f : N \to R$ surjective$\}$ (only possible if $n \ge r$).

$\#M(N,R)$, $\#Inj(N,R)$, $\#Surj(N,R)$?

Method is to reduce to counting of subsets of N or R.

$\#M(N,R)$: $1 \to$ (r ways to choose $f(1)$), $2 \to$ (r ways), $\dots$, $n \to f(n)$

By product formula:

$$\# \Rightarrow \underbrace{r \cdot r \cdots r}_{n \text{ times}} = r^n.$$

$\#Inj(N,R)$: $(f(1), \dots, f(n)) \subseteq \{1, \dots, r\}$ different & in order

$= P_{r}^{n}$ = an n-permutation of $\{1, \dots, r\}$

$\#Surj(N,R)$: To represent a surjective map $f : N \to R$ (maybe not surjective) is the same as to know about $f^{-1}\{1\}, f^{-1}\{2\}, \dots, f^{-1}\{r\} \subseteq \{1, \dots, n\}$

r subsets, maybe empty

$f^{-1}\{1\}, \dots, f^{-1}\{r\}$ are disjoint

& $\bigcup_{i=1}^{r} f^{-1}\{i\} = \{1, \dots, n\} = N$

f is surjective $\Leftrightarrow$ $f^{-1}\{i\} \neq \emptyset$ for every $i \in \{1, \dots, r\}$.

$\#Surj(N,R) \simeq$ ordered r-partitions of $\{1, \dots, n\}$.

$S_{n,r} =$ # of unordered r-partitions of $\{1, \dots, n\}$

Each element of $S_{n,r} \Leftrightarrow (r!)$ of ordered r-partitions.

Example: $n = 4, r = 2$ $\{1,2,3,4\}$ $\{3,4\}$ $\Leftrightarrow$ $\{1,2\} \sqcup \{3,4\}$, $\{3,4\} \sqcup \{1,2\}$

$$\# Surj(N,R) = r! \, S_{n,r}$$

(See previous paragraph for proof)

§1.3 Permutations

A cycle (cyclic permutation) of $\{1, \dots, n\}$ is something like $\begin{pmatrix} a_1 & a_2 & \cdots & a_{k-1} & a_k \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ a_2 & a_3 & & a_k & a_1 \end{pmatrix}$ all

$\Leftrightarrow$ corresponds to $f : \{1, \dots, n\} \to \{1, \dots, n\}$ such that

$$f(x) = \begin{cases} x & \text{if } x \notin \{a_1, \dots, a_k\} \\ a_{j+1} & \text{if } x = a_j,\ j \in \{1, \dots, k-1\} \\ a_1 & \text{if } x = a_k. \end{cases}$$

$\{1, \dots, 5\}$

$[1\ 3\ 4] = (3\ 2\ 4\ 1\ 5)$ ← previous notation

Theorem: Every permutation can be decomposed as a union of disjoint cycles in a unique way (up to the change order of the cycles).

Explain: Two cycles are disjoint if they have no common elements.

Example: $[1\ 2] \overset{\text{def}}{=} (2\ 1\ 3\ 4\ 5)$

$(f(1)\ f(2) \dots f(n))$ $[3\ 4\ 5] \overset{\text{def}}{=} (1\ 2\ 4\ 5\ 3)$

$[3\ 4] \overset{\text{def}}{=} (1\ 2\ 4\ 3\ 5)$

Then $[1\ 2]$ & $[3\ 4\ 5]$ are disjoint (does not have any of 3, 4 or 5)

$[1\ 2]$ & $[3\ 4]$ are disjoint

But $[3\ 4]$ & $[3\ 4\ 5]$ are not disjoint

Also $[3\ 4\ 5] = [4\ 5\ 3] = [5\ 3\ 4] = [3\ 4\ 5]$

We can shift to the right & get the same cycle

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2 disjoint cycles, when composed will give the same thing no matter the order.

A cycle = a permutation = a bijective map $f: \{1, \dots, n\} \to \{1, 2, \dots, n\}$.

If f_1 & f_2 are 2 permutations, then we can compose them:

1st way $f_1 \circ f_2$ $\{1, \dots, n\} \xrightarrow{f_2} \{1, \dots, n\} \xrightarrow{f_1} \{1, \dots, n\}$

2nd way $f_2 \circ f_1$ $\{1, \dots, n\} \xrightarrow{f_2} \{1, \dots, n\} \xrightarrow{f_1} \{1, \dots, n\}$

Example:

$(1\ 3\ 4\ 2)\ (2\ 4\ 3\ (1)) = ?$

$(2 \to 3)$ $(1 \to 2)$ $(3 \to 4)$ 1)

$(4 \to 2)$ $(2 \to 4)$

$(3 \to 4)$ $(3 \to 3)$

$(1 \to 1)$ $(4 \to 1)$

$(1\ 3\ 4\ 2)(2\ 4\ 3\ 1) = (3\ 2\ 4\ 1)$

$[1\ 3\ 4\ 2]$ $[1\ 2\ 4]$ $[1\ 3\ 4]$

$(2\ 4\ 3\ 1)(1\ 3\ 4\ 2) = ?$

$(2, 3, 1, 4)$

$(1 \mapsto 2)$ $(2 \mapsto 3)$ $(3 \mapsto 4)$ $(4 \mapsto 1)$

$(3 \mapsto 3)$ $(4 \mapsto 1)$ $(2 \mapsto 4)$ $(4 \mapsto 2)$

$(2\ 4\ 3\ 1)(1\ 3\ 4\ 2) = (2\ 3\ 1\ 4)$

$[1\ 2\ 4]\ [2\ 3\ 4] = [1\ 2\ 3]$

The composition of permutations is non commutative in general.

Lemma: If f_1 & f_2 are 2 disjoint cycles then $f_1 \circ f_2 = f_2 \circ f_1$ (so, commutative)

Proof: We need to show that for every $x \in \{1, \dots, n\}$ then $f_1 \circ f_2(x) = f_2 \circ f_1(x)$.

Notation: If f_1 is the cycle $(a_1 \to a_2 \to \dots \to a_k)$ then we write $\mathrm{Supp}(f_1) = \{a_1, \dots, a_k\}$.

Now f_1 & f_2 are disjoint $\Leftrightarrow \mathrm{Supp}(f_1) \cap \mathrm{Supp}(f_2) = \emptyset$.

If $x \notin \mathrm{Supp}(f_1) \cup \mathrm{Supp}(f_2)$:

$f_2(x) = x \Rightarrow f_1 \circ f_2(x) = x$

$f_1(x) = x \quad = f_2 \circ f_1(x)$

If $x \in \mathrm{Supp}(f_1) \Rightarrow \begin{cases} x \notin \mathrm{Supp}(f_2) \\ f_1(x) \notin \mathrm{Supp}(f_2) \end{cases}$

$\Rightarrow f_1(x) \in \mathrm{Supp}(f_1) \Rightarrow \begin{cases} f_2(x) = x \\ f_2 \circ f_1(x) = f_1(x) \end{cases}$

$f_1 \circ f_2(x) = f_1(x)$

$f_2 \circ f_1(x) = f_1(x)$

If $x \in \mathrm{Supp}(f_2) \Rightarrow$ similarly.