February 27, 2023

MAT 2250

§ 3.3. Generating functions of exponential type

$\{a_n\}_{n=1}^{\infty} \Rightarrow$ Generating function

$A(z) = a_0 + a_1 z + a_2 z^2 + \dots = \sum_{n=0}^{\infty} a_n z^n$

$\{a_n\}_{n=1}^{\infty} \Rightarrow$ Generating function of exponential type:

$\hat{A}(z) = a_0 + \frac{a_1}{1!} z + \frac{a_2}{2!} z^2 + \frac{a_3}{3!} z^3 + \dots = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$.

$\hat{A}(z) = B(z)$ where $B(z)$ is the generating function for $\{b_n = \frac{a_n}{n!}\}_{n=0}^{\infty}$ ($0! \stackrel{\text{def}}{=} 1$)

In some cases, then generating function of exponential type may be more easy to compute.

Proposition:

If $\hat{C}(z) = \hat{A}(z)\hat{B}(z)$,

then $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$, $\forall n$.

Proof:

$\hat{A}(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$

$\hat{B}(z) = \sum_{n=0}^{\infty} \frac{b_n}{n!} z^n$

$\hat{C}(z) = \sum_{n=0}^{\infty} \frac{c_n}{n!} z^n$

$\hat{A}(z)\hat{B}(z) = \sum_{n=0}^{\infty} d_n z^n$,

where $d_n = \sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!}$ $\forall n$

Since $d_n = \frac{c_n}{n!} \Rightarrow$

$\frac{c_n}{n!} = \sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!}$

$\Rightarrow$ Multiplying both sides by $n!$:

$c_n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a_k b_{n-k} \to \binom{n}{k}$. □

So if we see binomial coefficients, then may be generating functions of exponential type are helpful.

Example: (Exercise 3.14)

Find $\widehat{S}_m(z) = \sum_{n\geq 0} S_{n,m} \frac{z^n}{n!}$

$S_{n,m}$: Stirling number of second kind

Then show that

$$\sum_{m\geq 0} \widehat{S}_m(z)\, t^m = e^{t(e^z-1)}$$

From Exercise 1.35 we have:

$$S_{n+1,m+1} = \sum_{k=0}^{n} \binom{n}{k} S_{k,m} \qquad (*)$$

(Exercise 1.35 can be proven by induction, using the recurrence relation in formula (1.4).)

Fix m

$$\Rightarrow \widehat{S}_m(z) = \sum_{n=0}^{\infty} S_{n,m} \frac{z^n}{n!}.$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \widehat{B}(z) \text{ for } \{b_n = 1\}_{n=1}^{\infty}.$$

$$\widehat{B}(z)\,\widehat{S}_m(z) = \widehat{C}_m(z),$$

$$\Rightarrow e^z \widehat{S}_m(z) = \widehat{C}_m(z),$$

where $\widehat{C}_m(z)$ is the generating function of exponential type for $\{c_n\}_{n=1}^{\infty}$ with

$$c_n = \sum_{k=0}^{n} \binom{n}{k} \cdot 1 \cdot S_{k,m} \qquad (**)$$

Compare $(*)$ & $(**)$ $\Rightarrow c_n = S_{n+1,m+1}$

$$\widehat{C}_m(z) = S_{1,m+1} + \frac{S_{2,m+1}}{1!} z + \frac{S_{3,m+1}}{2!} z^2 + \cdots$$

$$= \left(\widehat{S}_{m+1}(z)\right)'$$

$$\widehat{S}_{m+1}(z) = S_{0,m+1} + \frac{S_{1,m+1}}{1!} z + \frac{S_{2,m+1}}{2!} z^2 + \cdots$$

$$\widehat{S}_{m+1}(z)' = e^z \widehat{S}_m(z).$$

Now

$$\widehat{\widehat{S}}(z,t) = \sum_{m=0}^{\infty} \widehat{S}_m(z)\, t^m$$

$$\Rightarrow e^z \widehat{\widehat{S}}(z,t) = \sum_{m=0}^{\infty} e^z \widehat{S}_m(z) t^m$$

$$= \sum_{m=0}^{\infty} S_{m+1}(z)' t^m$$

$$= \frac{d}{dz}\left(\sum_{m=0}^{\infty} S_{m+1}(z)\, t^m \right)$$

$$= \frac{1}{t} \frac{d}{dz}\left(\sum_{m=0}^{\infty} S_{m+1}(z)\, t^{m+1} \right)$$

$$= \frac{1}{t} \frac{d}{dz} \left(\hat{S}(z,t) - \hat{S}_0(z) t \right)$$

$$\hat{S}_0(z) = \sum_{n=0}^{\infty} \frac{S_{n,0}}{n!} z^n = 1$$

$$\left(S_{0,0} = 1, \quad S_{n,0} = 0 \text{ if } n > 0 \right)$$

$$e^z \hat{S}(z,t) = \frac{1}{t} \frac{d}{dz} \left(\hat{S}(z,t) \right) \quad (***)$$

Can check that $G(z,t) = e^{t(e^z - 1)}$ satisfies this $\hat{S}(z,t)$ (***). Also for $G(z,t)$ check initial condition, what is $S_{0,1}$? v.s. $\hat{S}(z,t)$

We know now (assuming you know how to solve ***) then:

$$\hat{S}(z,t) = \sum_{n,m \geq 0} \frac{S_{n,m}}{n!} z^n t^m \quad \text{(iv)}$$

$$= \boxed{e^{t(e^z - 1)}}$$

$e^{t(e^z-1)}$ by Taylor's expansion:

$$= 1 + \frac{t(e^z - 1)}{1!} + \frac{t^2 (e^z - 1)^2}{2!} \quad \text{(v)}$$

$$+ \cdots$$

$$e^z - 1 = \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

Example, if we want to know $S_{n,1}$, if we want $S_{n,2}$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{S_{n,m}}{n!} z^n = \frac{(e^z - 1)^m}{m!}.$$

Chapter 5: Asymptotic analysis

It helps to have a rough estimate of how large a sequence $\{a_n\}$ is, or how complex an algorithm is depending on the input.

Example

$$10^{10} \sim 10^{10} + 1000$$

1000 is too small compared to 10^{10}, so we can kind of ignore (vaguely).

§ 5.1 The growth of functions.

$f(n) \prec g(n)$ iff $\forall \varepsilon > 0 : \exists n_0 > 0$ so that $\forall n \geq n_0$: $|f(n)| \leq \varepsilon |g(n)|$.

Example:
$f(n) = n^9$
$g(n) = n^{10} - 10^{100} n^9 - 10^{1000} n^8 - 10^{10000} n^7$
then even though $f(n)$ is always positive for n positive, & $g(n)$ may be negative for example: $g(1) < -10^{10000}$, $f(n) \nless g(n)$.

O: $f(n) = O(g(n))$ iff
$\exists\, C > 0$ & n_0 : $|f(n)| \le C|g(n)|$
$\forall\, n > n_0$.

Same $f(n)$, $g(n)$
" n^9 " $n^{10} - 10^{100} n^9 - 10^{10000} n^8$
$- \cdots$
$\Rightarrow f(n) = O(g(n))$
& $n^{10} = O(g(n))$.

Ω: (opposite to O):
$f(n) = \Omega(g(n)) \Leftrightarrow$
$g(n) = O(f(n))$.

Θ (= $\Omega + O$):
$f(n) = \Theta(g(n)) \Leftrightarrow f(n) = O(g(n))$ & $f(n) = \Omega(g(n))$.
Θ is an equivalent relation.

$\sim$ (asymptotic equality):
$f \sim g \Leftrightarrow \lim_{n\to\infty} \frac{|f(n)|}{|g(n)|} = 1$.
($\sim$ is Θ with constants 1)

Example: $\left(1 - \frac{1}{n}\right)^n =$
$e^{-1}\left(1 - \frac{1}{2n} - \frac{5}{24n^2} + O\left(\frac{1}{n^3}\right)\right)$.
$\left(1 - \frac{1}{n}\right)^n = e^{n \log\left(1 - \frac{1}{n}\right)}$
$n \log\left(1 - \frac{1}{n}\right) = n\left[-\frac{1}{n} + \frac{\left(\frac{1}{n}\right)^2}{2} - \frac{\left(\frac{1}{n}\right)^3}{3} + \frac{\left(\frac{1}{n}\right)^4}{4} + \cdots\right]$

$= -1 + \frac{1}{2n} - \frac{1}{3n^2} + \frac{1}{4n^3} - \frac{1}{5n^4} + \cdots$
t
$e^{n \log\left(1 - \frac{1}{n}\right)} = e^{-1 + t} = e^{-1} \cdot e^t$
$= e^{-1}\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)$
We only care about $\frac{1}{n}$, $\frac{1}{n^2}$, so we can cut off at $\frac{t^2}{2!}$, & for t also cut off at $-\frac{1}{3n^2}$.
$\left[\log(1+t) = t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} \cdots\right]$

$O\left(\frac{1}{n^4}\right) = O\left(\frac{1}{n^3}\right)$

You may see the small o - notation.

Small o = $<$.

Why should we care about these notations? Because if we really want to compute some numbers which are either too big, too small or not in $\mathbb{Q}$, then we usually need to use some infinite series, & our computers (& our selves) cannot work with infinite, so we need to know where to cut off to work with a finite sum!