1.8

First we show the inequalities on the left hand side. Let $k_1 < k_2 \leq \lfloor n/2 \rfloor$, it suffices to show that $\binom{n}{k_1} < \binom{n}{k_2}$. This is equivalent to

$$\frac{n!}{k_1!(n-k_1)!} < \frac{n!}{k_2!(n-k_2)!} \Leftrightarrow \frac{1}{k_1!(n-k_1)!} < \frac{1}{k_2!(n-k_2)!} \Leftrightarrow \frac{k_2!}{k_1!} < \frac{(n-k_1)!}{(n-k_2)!}.$$

The last inequality holds because the two sides are products of the same number of terms (one for each integer greater than k_1 and less than or equal to k_2), each of which is larger on the right by our assumption $k_1 < k_2 \leq \lfloor n/2 \rfloor$.

That $\binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} > \cdots > \binom{n}{n}$, follows from the symmetry $\binom{n}{k} = \binom{n}{n-k}$ and the argument above.

(Explanation of) 1.10

For each n to have a *unique* expression $n = \sum a_k k!$ with $0 \leq a_k \leq k$, first we have argue why any n has *at least one* such expression; one way to do this is to provide an algorithm that finds coefficients a_k. Based on the requirement $0 \leq a_k \leq k$, the leading coefficient a_m must be indexed by the (unique) m such that $m! \leq n < (m+1)!$, choosing then a_m maximal with respect to the condition that $n \geq a_m m!$ ensures that $n - a_m m! < m!$ (otherwise we could increase a_m).

The existence part of the "inducting on m" in the book's solution means we determine the coefficients a_k by repeating the above for $n - a_m m!$ to find the next nonzero coefficient, say $a_{m'}$. The use of the term induction in this sense is a slight abuse, referring to the fact that we are defining the expression recursively.

The uniqueness part is more like usual induction, but instead we induct on n (book says m), i.e. assuming integers less than n have such a unique expression can we show that n does (this is *strong induction*) (the base cases $n = 0, 1$ are clear, if we correct the sum to be indexed over $k \geq 1$ (otherwise $1 = 1 \cdot 0! = 1 \cdot 1!$)). The process above determines the first term in the expression uniquely, and since $n - a_m m!$ has a unique expression by the inductive hypothesis, so does n.