### 2.2

Compute $f(x) = \sum_{k=0}^{n} kx^k$:

(a) "Isolation of terms".

$$\begin{aligned} f(x) = \sum_{k=0}^{n} kx^k &= x + 2x^2 + 3x^4 + \ldots + nx^n \\ &= x(\sum_{k=0}^{n-1}(k+1)x^k) = x(\sum_{k=0}^{n-1} kx^k + \sum_{k=0}^{n-1} x^k) \\ &= x((\sum_{k=0}^{n} kx^k) - nx^n + \frac{x^n - 1}{x-1}) \\ &= x(f(x) - nx^n + \frac{x^n - 1}{x-1}) \end{aligned}$$

Thus,

$$f(x) - xf(x) = (1-x)f(x) = -nx^{n+1} + \frac{x(x^n-1)}{x-1}$$

$$(x-1)f(x) = nx^{n+1} - \frac{x(x^n-1)}{x-1}$$

$$f(x) = \frac{nx^n(x-1) - x(x^n-1)(x-1)}{(x-1)^2} = \frac{x(nx^{n+1} - (n+1)x^n + 1)}{(x-1)^2}$$

(c) "By differentiating" Let $g(x) = \sum_{k=0}^{n} x^k$. Then

$$\frac{d}{dx} g(x) = \sum_{k=0}^{n} kx^{k-1} = f(x)/x.$$

Also $g(x) = \frac{x^{n+1}-1}{x-1}$ and by the quotient rule

$$\frac{d}{dx} g(x) = \frac{(x-1)((n+1)x^n) - (x^{n+1}-1)}{(x-1)^2} = \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}.$$

### 2.7b

Take $u(k) = 2k+1$ and $\Delta v(k) = 1/k(k+1)$, so $v(k) = -1/k$. See stackexchange.

### 3.1

One way to see the identity using generating functions is to see both sides as the same coefficient (say the $n$-th) coefficient of two expressions of the same generating function. We know that $\binom{2n}{n}$ is the $n$-th coefficient of $(1+z)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} z^k$. Using the "convolution product" for $((1+z)^n)^2 = (\sum_{k=0}^{n} \binom{n}{k} z^k)^2$ we get

$$(1+z)^{2n} = \sum_{m=0}^{2n} \sum_{k=0}^{m} \binom{n}{k} \binom{n}{m-k} z^m.$$

The $n$-coefficient of this expression is $\sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2$.