5.3

We're considering

$$f(n) = \begin{cases} n^2 & n \text{ even} \\ 2n & n \text{ odd} \end{cases}.$$

Since $n^2 > 2n$ for all $n > 2$ it follows that $f(n) = O(n^2)$ (can take the constant C in the definition to be 1.) To show that $f(n) \neq o(n^2)$, we need to show that there exists $\varepsilon > 0$ such that for all $n_0(\varepsilon)$ there is $n \geq n_0(\varepsilon)$ with $f(n) > \varepsilon |n^2|$; take $\varepsilon < 1$ and n even. Finally $n^2 \neq O(f(n))$: for any choice of constant C, if n odd $> 2C$ (for example $n = 2C + 1$) we have $n^2 > C \cdot 2n$.

5.5

For each $f(n)$ we need to find a function $g(n)$ such that $f(n) = O(g(n))$, but $g(n) \neq \Theta(f(n))$. Since $g(n) \neq \Theta(f(n))$ if and only if $f(n) \neq \Theta(g(n))$, and $f(n) \neq \Theta(g(n))$ if and only if $f(n) \neq O(g(n))$ or $f(n) \neq \Omega(g(n))$, the problem can be re-stated as find $g(n)$ with $f(n) = O(g(n))$ and $f(n) \neq \Omega(g(n))$ (equivalently $g(n) \neq O(f(n))$.

(a) $f(n) = \binom{n}{2} = \frac{1}{2}n(n-1)$. Take $g(n) = n^3$.
(b) $g(n) = n^3$.
(c) $g(n) = n^2 3^n$.
(d) $g(n) = n^n$ (see Stirling's approximation).

6.6

Let m be the maximal length. Suppose there are two paths of maximal length $u_1, \ldots, u_m$ and $u_1, \ldots, u_m$ that don't have a common vertex (i.e. $u_i \neq v_j$ for any i, k). Since we assume that the graph is connected there must be some path $u_i, \cdots v_j \, for \, some \, i, j$. Without loss of generality assume that $i \geq m/2 \geq j$ (otherwise we can reverse the order of the path). We now create a new path, $u_1, \ldots, u_i, \ldots v_k, \ldots v_m$ which has length $\geq m + 1$, contradicting the maximality of m.