6.2

First some observations:

1. The maximum possible degree in a graph with n vertices is $n-1$. If a graph has a vertex with this degree, then there are no vertices with degree 0.
2. Given any n-graph (graph with n vertices) satisfying the condition, one can form another graph satisfying the condition by taking its *complement* $\tilde{G}$: If two vertices share an edge in G then they do not in $\tilde{G}$ and vice versa. Alternatively, G can be viewed as a subgraph of the complete graph K_n and $\tilde{G} = K_n \setminus G$. The complement graph will also have $n-1$ different degrees, namely $\{n - d_i\}$ where d_i are the degrees of the vertices in the original graph.
3. Any graph satisfying the condition has either a vertex of degree $n-1$ or a vertex of degree 0 (otherwise there are only $n-2$ possible degrees).
4. A graph with $n \geq 3$ vertices satisfying the condition cannot have *two* vertices of degree zero. Otherwise removing both vertices would create an $(n-2)$-graph with at least $n-2$ different degrees, contradicting 1.

Let G be a graph with n vertices having $n-1$ different degrees. By (3) it has either a vertex with degree zero or a vertex with degree $n-1$. In the later case we consider the complement graph (see (2)). By deleting the unique vertex with degree 0 (observation 4), we can form an $(n-1)$-graph that has $(n-1)-1$ different degrees.

This shows that all such graphs will be the G_i and their complements $\tilde{G}_i$ obtained by the following algorithm:

1. Let G_2 be the graph with two vertices not sharing an edge.
2. $\tilde{G}_i$ is the complement of G_i
3. $G_{i+1} = \tilde{G}_i$ + a new disjoint vertex.

6.10

We can rewrite the inequality as $\alpha(G) \cdot \Delta \geq |V| - \alpha(G) = |V \setminus A|$, so it suffices to show why $\alpha(G) \cdot \Delta$ gives an upper bound on the number of vertices not in A. This would be maximized if each vertex in A had maximal degree Δ, and each vertex outside V was connected to at most one vertex in A. By the maximality of A, every vertex outside in $V \setminus A$ shares a vertex with something in A. Therefore $|V \setminus A| \leq \alpha(G) \cdot \Delta$.

6.14

Show that G has no cut edges if all its degrees are even.

It suffices to show that any edge, say e, is contained in a circuit. Since every vertex has even degree we can pair off incident edges at a vertex. Extend the path given by e on each side (at each vertex) by adding the the other edge in the pair. Continue extending the path in this way, since the graph is finite a circuit must be formed.

6.19

Show that $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} \min(d_j, k)$ *for all* k.

Label the vertices according to their degree so that v_1 has degree d_1. We want to realize $k(k-1) + \sum_{j=k+1}^{n} \min(d_j, k)$ as an upper bound for the sum of all degrees of the "first" k vertices $v_1, \ldots, v_k$. Recall that the degree of a vertex is the number of edges incident to that vertex, we break this sum up into contributions coming from edges between the first k vertices and those coming from connections to $v_{k+1}, \ldots v_n$. The first part is maximized when each of the k vertices in $v_1, \ldots v_k$ are connected to all the others, contributing $k(k-1)$ to the total degree of theses vertices. The second part, (counting edges from the $v_{k+1}, \ldots v_n$ to $v_1, \ldots v_k$) is bounded by $\sum_{j=k+1}^{n} \min(d_j, k)$ since the vertex v_j with degree d_j could be connected to all the first k vertices, or if $d_j < k$ every neighbor of v_j could be within $v_1, \ldots v_k$ (contributing d_j).