MAT2250 Exercise list
Spring 2024 - Instructor: Kris Shaw

10. Week of April 15th - Latin squares and designs

Warm ups.

(1) How many mutually orthogonal Latin squares are there of order 5? Find a pair of them.
(2) Why is there at least one pair of orthogonal Latin squares of order 12?
(3) Write down the blocks $B_1, \ldots, B_7 \subset \{1, \ldots, 7\}$ for the 2-block design from the Fano plane.
(4) How many lines and points are in a finite projective plane of order 3. How many points are on each line?

Problem 10.1. Show that permuting the rows, columns, or symbols preserves the orthogonality of Latin squares.

Problem 10.2. Construct a pair of orthogonal Latin squares of order 12.

Problem 10.3. Let $D(v, k, \lambda)$ be a 2-design with b blocks $B_1, \ldots, B_b$ on $\{1, \ldots, v\}$ and $M = (m_{ij})$ the $v \times b$ matrix given by $m_{ij} = 1$ if $i \in B_j$ and $m_{ij} = 0$ if $i \notin B_j$. Compute MM^T and its determinant and show that it is non-zero. Conclude that $v \leq b$. Hint: Use that $rank(MM^T) \leq rank(M)$.

Problem 10.4. Show that there exists a finite projective plane of order n if and only if there exist $n-1$ orthogonal latin squares of order n.
(Hint: Consider two points x, y of the plane and the line L passing through them. Enumerate the other lines through x by $L_1, \ldots L_n$ and the lines through y by $J_1, \ldots, J_n$ (draw a cartoon picture). Think about the n^2 points of the plane that are not contained on L as the n^2 entries of a latin square where the point p_{ij} corresponds to entry (i, j) if p_{ij} is on lines L_i and J_j. The L line contains $n-1$ other points and there are exactly n lines through each of these points. Use each of these $n-1$ points to come up with a latin square so the $n-1$ squares are mutually orthogonal.)

Problem 10.5. Let P be a finite projective plane of order q. Any three collinear points form a triangle. Show that the number of triangles is $\frac{1}{6}q^3(q+1)(q^2+q+1)$.