MAT2250 Exercise list
Spring 2024 - Instructor: Kris Shaw

2. Week of January 22nd - Recurrence relations and generating functions

Warm ups.

(1) Expand out the rising factorial polynomials $x^{\overline{3}}$ and $x^{\overline{4}}$ and identity the coefficients as Stirling numbers of the first kind.
(2) Use the recurrence relation for binomial coefficients to give a recurrence relation for falling factorials.
(3) Let (a_n), (b_n) and (c_n) be sequences indexed by $n \in \mathbb{N}$ and let $A(z), B(z)$, and $C(z)$ be their respective generating functions. What are the generating functions for the following sequences in terms of $A(z)$, $B(z)$, $C(z)$ and some known generating functions? (sums, differences, products, derivatives, and substitutions are all allowed!)
   (a) $(a_n + 10b_n - c_n)$
   (b) $(2^n a_n)$
   (c) $(a_n + b_{n-1})$ (assume $b_{-1} = 0$)
   (d) $(a_n + 1)$
   (e) (na_n)
(4) Find the sequences given by the generating functions:
   (a) $\frac{x}{1-2x}$
   (b) $\frac{1}{(1-x)^2}$ Hint: What is the derivative of $\frac{1}{1-x}$?
   (c) $\frac{1}{1+x+y}$ Hint: the coefficient $x^k y^{n-k}$ is one of beloved counting coefficients

Problem 2.1. Prove the following recurrence formula for Stirling numbers of the second kind:

$$S_{n+1,k+1} = \sum_{i=k}^{n} \binom{n}{i} S_{i,k}$$

Hint: empathise with the element $n+1$ and partition the set of all (k+1)-set partitions of $\{1, \ldots, n+1\}$ based on the size of the subset in the partition containing $n+1$.

Prove the following recurrence formula for the Stirling numbers of the first kind:

$$\sigma_{n+1,k+1} = \sum_{i=k}^{n} \binom{i}{k} \sigma_{n,i}$$

Hint: empathise with the element $n+1$ and partition the set of permutations of $\{1, \ldots, n+1\}$ with k-cycles based on the length of the cycle containing $n+1$.

Problem 2.2. Let $P_{n,k}$ be the number of integer partitions of n with k parts.

(1) Give a combinatorial proof of the following relation for integer partitions:

$$P_{n,k} = \sum_{i=1}^{k} P_{n-k,i}.$$

What initial conditions need to be fixed in order to make this a recurrence relation giving all numbers $P_{n,k}$?

Hint: Remove the first column from the Young diagram of a partition of n with k parts and count what remains.

(2) Use the fact that every positive integer can be uniquely expressed as a sum of distinct powers of two to prove the following identity:

$$\frac{1}{1-x} = \prod_{k=1}^{\infty}(1 + x^{2^k}).$$

Problem 2.3. Let a_n be the number of ways a $2 \times n$ rectangle can be covered by 1×2 dominoes. Determine a recurrence relation for the sequence (a_n) and find its generating function $A(z) = \sum_{n=0}^{\infty} a_n z^n$ as a rational function.

Let b_n be the number of ways a $3 \times n$ rectangle can be covered by 1×3 dominoes. Determine a recurrence relation for (b_n) and find its generating function $B(z) = \sum_{n=0}^{\infty} b_n z^n$ as a rational function.

Problem 2.4. Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$. Show that $\frac{1}{2}(A(z)+A(-z)) = \sum_{n=0}^{\infty} a_{2n} z^{2n}$ and $\frac{1}{2}(A(z)-A(-z)) = \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1}$.