MAT2250 Exercise list
Spring 2024 - Instructor: Kris Shaw

3. Week of January 29th - Generating functions continued

Warm ups.

(1) Let the sequence a_n be defined by the recurrence $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ and $a_0 = 1$, $a_1 = 1$, $a_2 = 1$. Find an expression of the generating function for a_n as a rational function (do not find the partial fraction expansion).
(2) Suppose the generating function of a sequence a_n is given by $A(z) = \frac{1}{(1-z)^2} + \frac{2}{1-2z}$
(3) What is the number of partitions of 7 into distinct parts? What is the number of partitions of 7 into odd parts?
(4) Find the closed form for the exponential generating function for sequence given by the number of permutations of a set of size n.

Problem 3.1. Determine the recurrence relations, generating functions, and an explicit formula for the following sequences in n:

(1) The number of words of length n on an alphabet of size 5.
(2) The number of binary words of length n with no consecutive ones.
(3) The number of binary words of length n with an even number of zeros.
(4) The number of words on the alphabet $(0, 1, 2)$ with an even number of zeros and an even number of ones.

Problem 3.2. Recurrences for Stirling numbers of the second kind

(1) Use a counting argument to show that the Stirling numbers of the second kind satisfy $S_{n,2} = 2^{n-1} - 1$ for $n \geq 1$ and use this to find an expression for the generating function
$$\mathcal{S}_2(z) = \sum_{n=o}^{\infty} S_{n,2} z^n.$$
(2) Use the recurrence relation $S_{n,k} = S_{n-1,k-1} + kS_{n-1,k}$ and the formula found for $\mathcal{S}_2(z)$ to show
$$\mathcal{S}_3(z) := \frac{z^3}{(1-z)(1-2z)(1-3z)}$$
then use partial fractions to find a closed formula for $S_{n,3}$.

Problem 3.3. Let $P_n^{1,2}$ be the number of partitions of n using only parts of size 1 and 2 with $P_0^{1,2} := 1$.

(1) Calculate by hand $P_n^{1,2}$ for $n \leq 5$.
(2) Find the generating function for the sequence $P_n^{1,2}$.
(3) Give an explicit formula for $P_n^{1,2}$. *Hint: Use partial fraction decompositions to express the generating function from above in the form*

$$\frac{A}{1-z} + \frac{B}{(1-z)^2} + \frac{C}{1+z}.$$

Problem 3.4. Let T_n be the number of ways of triangulating a labeled $(n+2)$-gon (see end of the Catalan numbers screencast). Set $T_0 := 1$. Explain a recurrence formula for T_n and conclude that $T_n = C_n$ where C_n is the n-th Catalan number. *Hint: In a triangulation consider the triangle that has vertices, 1, 2, k (where k is determined by the triangulation) and think about "removing" this triangle like we did with the bracket expressions.*

Problem 3.5. Exponential generating functions

(1) Find the generating function and the exponential generating function for the sequence $a_n = 2^n + 5^n$.
(2) Recall the Bell numbers from Week 2 Problem 1.3. Using the recursive formula proven for $\tilde{B}_{n+1}$ in Problem 1.3, show that the exponential generating function $\hat{B}(z)$ for $\tilde{B}_{n+1}$ satisfies:

$$\frac{d\hat{B}(z)}{dz} = e^z \hat{B}(z).$$

Verify that $\hat{B}(z) = e^{e^z - 1}$ satisfies the above differential equation and that the first 3 terms of the series expansion of e^{e^z-1} give $\frac{\tilde{B}_i}{i!}$ for $i = 0, 1, 2$.