MAT2250 Exercise list
Spring 2024 - Instructor: Kris Shaw

4. Week of February 5th - Inversion, Inclusion-Exclusion

Warm ups.

(1) What is the number of permutations on the set $\{1, \ldots, 4\}$ with k-fixed points for $k = 0, 1, 2, 3, 4$?
(2) Apply binomial inversion to the sequence $u_k = 1$ for all k, what is v_k? Now use the binomial inversion formula for u_n in terms of v_k, and confirm the result.
(3) Apply the binomial inversion to the sequence $v_k = 1$ for all k, what is u_k?
(4) Explain why the following is equivalent to binomial inversion:

$$v_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} u_k \iff u_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} v_k.$$

(5) Use inclusion-exclusion to find how many $0, 1, 2$ words of length 5 contain each number at least once.
(6) How many numbers less than 6 are relatively prime to 6? How many numbers less than 42 are relatively prime to 42?

Problem 4.1. Recall that Bell numbers $\tilde{B}_n$ count the number of partitions of a set of size n. Use an inversion formula to prove that:

$$\sum_{k=0}^{n} (-1)^{n-k} \sigma_{n,k} \tilde{B}_k = 1.$$

Check the formula for $n \leq 4$. Convince yourself that it is a striking formula without an immediate combinatorial proof!

Problem 4.2. The Lah numbers are defined by

$$L_{n,k} = (-1)^n \frac{n!}{k!} \binom{n-1}{k-1},$$

for $n, k \geq 0$. Prove the Lah inversion formula

$$v_n = \sum_{k=0}^{n} L_{n,k} u_k \iff u_n = \sum_{k=0}^{n} L_{n,k} v_k.$$

Hint: prove that

$$(-x)^{\underline{n}} = \sum_{k=0}^{n} L_{n,k} x^{\underline{k}}$$

by using a Vandermonde identity for $\binom{x+n-1}{n}$. How can you use the same identity to write $x^{\underline{n}}$ in terms of $(-x)^{\underline{k}}$?

Problem 4.3. Prove using inclusion-exclusion that the number of surjective maps from $\{1,\dots,n\}$ to the set $\{1,\dots,r\}$ is

$$\sum_{i=0}^{r}(-1)^i\binom{r}{i}(r-i)^n.$$

Deduce that $S_{n,r}=\sum_{i=0}^{r}\frac{(-1)^i(r-i)^{n-1}}{i!}$.

Problem 4.4. Let $\phi(n)$ denote the number of integers k such that $1\leq k\leq n$ and with k relatively prime to n. Prove using inclusion-exclusion that $\phi(n)=n(1-\frac{1}{p_1})\dots(1-\frac{1}{p_t})$, where $n=p_1^{l_1}p_2^{l_2}\dots p_t^{l_t}$ for distinct primes $p_1,\dots,p_t$ and $l_i>0$ for all $1\leq i\leq t$.

Problem 4.5. Generalisation of Inclusion-Exclusion

(1) Let $E_1,\dots E_m$ be properties of elements of an n-set S and let $N(E_{i_1}\dots E_{i_t})$ denote the number of elements of S that satisfy the properties $E_{i_1},\dots,E_{i_t})$ (and possibly more).

Show that the number of elements that satisfy exactly t properties is given by

$$\sum_{i_1<\dots<i_t}N(E_{i_1}\dots E_{i_t})-\sum_{i_1<\dots<i_{t+1}}\binom{t+1}{t}N(E_{i_1}\dots E_{i_{t+1}})+\dots(-1)^{m-t}\binom{m}{t}N(E_1\dots E_m).$$

(2) Let N_r be the number of elements of S that satisfy at least r of the properties $E_1,\dots,E_m$ (in other words $N_r=\sum_{i_1<\dots<i_r}N(E_{i_1}\dots E_{i_r})$). Let G_r be the number of elements of S that satisfy exactly r of the properties $E_1,\dots,E_m$.

Relate the generating functions $N(z)=\sum_{r\geq 0}N_rz^r$ and $G(z)=\sum_{r\geq 0}G_rz^r$.