MAT2250 Exercise list
Spring 2024 - Instructor: Kris Shaw

5. Week of February 12th - Asymptotic analysis

Warm ups.

(1) Let $f_1(n) = n$, $f_2(n) = n^2$, $g_1(n) = n^3 + n$ and $g_2(n) = -n^3$. Show that $f_1(n) \prec g_1(n)$, $f_2(n) \prec g_2(n)$, but that we don't have that $f_1(n) + f_2(n) \prec g_1(n) + g_2(n)$.
(2) Find an example of two polynomials $f(n)$ and $g(n)$ such that $f(n) < g(n)$ but that we do not have $f(n) \prec g(n)$.
(3) If an algorithm is such that for an input of length n executes n steps where the i-th step requires 2 operations what is the big-O running time of the algorithm?
(4) If an algorithm is such that for an input of length n executes n steps where the i-th step requires n^2 operations what is the big-O running time of the algorithm?

Problem 5.1. Show that $n^n < (n!)^2$ for $n \geq 3$. Moreover use Stirling's formula for the asymptotics of $n!$ to show that $n^n \prec (n!)^2$.
Hint: Show that when $n \geq 3$, we have $n < k(n-k+1)$ for all $3 \leq k \leq n$.

Problem 5.2. Use Stirling's formula to estimate the growth of $\binom{n}{\frac{n}{2}}$ for even n.

Problem 5.3. Prove the following equalities of sets:

(1) $O(f(n))O(g(n) = O(f(n)g(n))$
(2) $O(f(n)g(n)) = f(n)O(g(n))$
(3) $O(f(n) + O(g(n)) = O(|f(n)| + |g(n)|)$. Is this statement true if the absolute value bars are removed from the right hand side?

Problem 5.4. The following method of Strassen's can be used to compute the product of two 2×2 matrices using only 7 multiplications, instead of the 8 multiplications required in the usual method.
Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Show that each element of the product AB can be written as a sum of terms $\pm m_i$ where,

$$m_1 = (a+d)(\alpha+\delta), \qquad m_2 = (c+d)\alpha, \qquad m_3 = a(\beta-\delta), \qquad m_4 = d(\gamma-\alpha),$$

$$m_5 = (a+b)\delta, \qquad m_6 = (a-c)(\alpha+\beta), \qquad m_7 = (b-d)(\gamma+\delta).$$

How many additions and subtractions are required in the usual method versus Strassen's method?