MAT2250 Exercise list
Spring 2024 - Instructor: Kris Shaw

4. Week of February 18th - Intro to graph theory

Warm ups.

(1) How many edges are in the complete graph on n vertices?
(2) Is there a 3 regular graph on 5 vertices?
(3) Write down the adjacency and incidence matrices for the complete graph on 4 vertices.
(4) Write down the adjacency and incidence matrices for a labelling of the circuit graph on 4 vertices.
(5) Write down the adjacency and incidence matrices for the hypercube graph Q_2.
(6) How many orientations are there of a graph G with q edges?
(7) How many acyclic orientations are there of K_3?

Problem 4.1. Show that a graph with $n \geq 2$ vertices has at least two vertices of the same degree. Hint: the degree of a vertex must be less than or equal to $n-1$. If there is a vertex of degree $n-1$ can there be a vertex of degree 0?

Problem 4.2. Show that for every even integer $n \geq 4$ there is a 3-regular graph on n vertices. Hint: What is the graph when $n = 4$. Given a 3-regular graph with n vertices, prove that there exists a pair of non-incident edges. How can you connect two new degree 3 vertices to the graph to obtain a 3-regular graph with $n+2$ vertices?

Problem 4.3. A k-colouring of a graph $G = (V, E)$ is an assignment $f : V \to C$ where $|C| = k$ and such that $uv \in E$ implies $f(u) \neq f(v)$ (imagine that C is considered a set of colours).
Determine the number of k-colorings of the complete graph K_n, the path graph P_n, and the circuit graph C_n.

Problem 4.4. A *cut edge* k of a graph $G = (V, E)$ is an edge such that the graph $(V, E \backslash k)$ has more components than G. Show that an edge k is a cut edge if and only if it is contained in no circuits. Can you describe graphs that only have cut edges?
A *cut vertex* v of a graph $G = (V, E)$ is a vertex such that removing it and all incident edges increases the number of components. Show that a connected graph with exactly two non-cut vertices is a path.

Problem 4.5. Let $G = (V, E)$ be a graph. For $k, k' \in E$, define $k \sim k'$ if either $k = k'$ or k, k' lie on a circuit. Show that $\sim$ is an equivalence relation. Is it the same equivalence relation from the screen cast? Show that the edges of a graph G can be oriented in such a way that the resulting directed graph $\vec{G}$ is strongly connected if an only if G is connected and has no cut edges. Hint: Use induction on the number of $\sim$ equivalence classes of edges of G.