MAT2250 Exercise list
Spring 2024 - Instructor: Kris Shaw

6. Week of February 26th - Trees

Warm ups.

(1) Draw the spanning tree of K_7 of the Prüfer code $(1, 1, 3, 2, 6)$.
(2) Find the Prüfer code of the path P_n labelled sequentially with $1, 2, \ldots, n$.
(3) How many edges must be removed from the hypercube graph to obtain a spanning tree?
(4) What trees do the breadth first and depth first searches return on K_n?
(5) What are the bases of the matroid of the circuit graph C_n?

Problem 6.1. Let $d_1, \ldots, d_n$ be natural numbers with $d_i \geq 1$ and $\sum_{i=1}^{n} d_i = 2n-2$. Show that the number of spanning trees of K_n with vertex i having degree d_i is equal to

$$\frac{(n-2)!}{(d_1-1)!\ldots(d_n-1)!}.$$

Problem 6.2. Perform the depth first search and breadth first search algorithms on the hypercube graph Q_3 from starting vertex $(0, \ldots, 0)$ and describe the resulting trees.

Problem 6.3. Label the vertices of K_n with $\{1, \ldots, n\}$ and consider the weight function given by $w(ij) = i + j$. Construct a minimal spanning tree T for this weight function. Is this tree unique? What is $w(T)$?

Problem 6.4. Let $(S, \mathcal{U})$ be a collection of sets satisfying axioms 1 and 2 of a matroid. Let $w : S \to \mathbb{R}$ be a weight function on S. The greedy algorithm applied to $(S, \mathcal{U})$ with weight function w selects at each step a subset $A_i \in \mathcal{U}$ such that $A_0 = \emptyset$, $A_{i+1} = A_i \cup a_{i+1} \in \mathcal{U}$ for $a_{i+1} \notin A_{i+1}$ and ends at A_r when there is no subset $A' \in \mathcal{U}$ such that $A_r \subsetneq A'$.

Show that $(S, \mathcal{U})$ satisfies the 3rd matroid axiom 3 as well if and only if the greedy algorithm yields the optimum for every weight function $w : S \to \mathbb{R}$.
Hint: For $A, B \in \mathcal{U}$, with $|B| = |A| + 1$, consider $w : S \to \mathbb{R}$ given by $w(x) = -|A| - 2$ for $x \in A$ and $w(x) = -|A| - 1$ for $x \in B \backslash A$ and $w(x) = 0$ otherwise. Derive a contradiction.