MATHEMATICAL LOGIC

In a nutshell, mathematics consists of definitions, theorems and proofs.

Definition The introduction of a new term.

Example: "A number is <u>even</u> if it is divisible by 2"

Theorem A mathematical, provable fact

Example: "There exists infinitely many even numbers"

Proof (NO: bevis) A logical argument for the validity of a theorem.

Lemma A small theorem aimed to prove a bigger theorem

Proposition (NO: Proposisjon) A small theorem of independent interest

Corollary (NO: Korollar) A theorem which follows (more or less directly) from another theorem

To state theorems, definitions and proofs, we will use the following symbols:

$\boxed{\Longrightarrow}$ "implies" (NO: "impliserer"). "$A \Longrightarrow B$" means "if A is true then B is true".

Example: n is an even number $\Longrightarrow$ n^2 is an even number

$\boxed{\Longleftarrow}$ "is implied by". "$A \Longleftarrow B$" is the same as "$B \Longrightarrow A$".

$\boxed{\Longleftrightarrow}$ "is equivalent to", or "if and only if" (NO: "er ekvivalent med", "hvis og bare hvis").

"$A \Longleftrightarrow B$" means <u>both</u> $A \Longrightarrow B$ and $A \Longleftarrow B$.

Example: n is an even number $\Longleftrightarrow$ n^2 is an even number

$\forall$ "for all" (NO: "for alle")

Example: n^2 is even $\forall$ even n

$\exists$ "there exists" (NO: "det eksisterer")

Example: If x is a number then $\exists$ a number $y > x$

$\exists!$ "there exists a unique" (NO: "det eksisterer en unik")

Example: If $x > 0$ then $\exists!$ $y > 0$ such that $y^2 = x$

$:$ "such that"

Methods of proof

In this course you will see a lot of proofs!

Some of the most common strategies are:

a) proof by induction

b) proof by contradiction

c) proving the contrapositive

a) Proof by induction

Claim: Some statement $P(n)$ is true for every $n \in \mathbb{N}$.

Strategy:

- First prove $P(1)$
- Then prove that $P(n) \Rightarrow P(n+1)$, for any $n \in \mathbb{N}$.

Proposition: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for any $n \in \mathbb{N}$.

Proposition: $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for any $n \in \mathbb{N}$.

Proof: • We have $\displaystyle\sum_{k=1}^{1} k = 1$ and $\frac{1\cdot(1+1)}{2} = 1$, so $P(1)$ is true.

• Assume $P(n)$ is true for some $n \in \mathbb{N}$. We have

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1)$$

$$= \frac{(n+1)(n+2)}{2}$$

so $P(n+1)$ is also true. $\blacksquare$

b) Proof by contradiction (reductio ad absurdum)

Strategy: To prove P, assume (not P), and reach a contradiction.

Proposition: $\sqrt{2}$ is irrational

Proof: Assume the converse – that $\sqrt{2}$ is rational.

Let $p, q \in \mathbb{N}$ be such that $\sqrt{2} = p/q$.

We may assume that p and q have no common prime factors.

Then $2 = p^2/q^2 \iff p^2 = 2q^2$. Thus, p is even: $p = 2r$.

Therefore, $4r^2 = 2q^2 \iff q^2 = 2r^2$, so q is also even.

But p and q have no common prime factors! $\lightning$ $\blacksquare$

c) Proving the contrapositive

Strategy: To prove $P \Rightarrow Q$, prove $(\text{not } Q) \Rightarrow (\text{not } P)$

Note: To prove $P \Leftrightarrow Q$, it is common to prove $P \Rightarrow Q$ and $(\text{not } P) \Rightarrow (\text{not } Q)$

Proposition: If $n \in \mathbb{N}$ is such that the sum of its positive divisors is $n+1$, then n is prime

Proof: Let $n \in \mathbb{N}$ be non-prime. Then it is divisible by 1, n, and at least one more number $x \in \mathbb{N}$. Hence,

$$(\text{sum of positive divisors}) \geq 1 + n + x > n + 1.$$

$\blacksquare$

QUESTIONS ?

COMMENTS ?