

MATHEMATICAL LOGIC

In a nutshell, mathematics consists of definitions, theorems and proofs.

Definition

The introduction of a new term.

Example: "A number is even if it is divisible by 2"

Theorem

A mathematical, provable fact

Example: "There exists infinitely many even numbers"

Proof

(NO: bevis) A logical argument for the validity of a theorem.

Lemma

A small theorem aimed to prove a bigger theorem

Proposition

(NO: Proposition) A small theorem of independent interest

Corollary

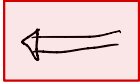
(NO: Korollar) A theorem which follows (more or less directly) from another theorem

To state theorems, definitions and proofs, we will use the following symbols:



"implies" (NO: "implieser"). " $A \Rightarrow B$ " means "if A is true then B is true".

Example: n is an even number $\Rightarrow n^2$ is an even number



"is implied by". " $A \Leftarrow B$ " is the same as " $B \Rightarrow A$ ".



"is equivalent to", or "if and only if" (NO: "er equivalent ned", "hvis og bare hvis").

" $A \Leftrightarrow B$ " means both $A \Rightarrow B$ and $A \Leftarrow B$.

Example: n is an even number $\Leftrightarrow n^2$ is an even number

\forall

"for all" (NO: "for alle").

Example: n^2 is even \forall even n

\exists

"there exists" (NO: "det eksisterer")

Example: If x is a number then \exists a number $y > x$

$\exists!$

"there exists a unique" (NO: "det eksisterer en unik")

Example: If $x > 0$ then $\exists!$ $y > 0$ such that $y^2 = x$

$:$

"such that"

Methods of proof

In this course you will see a lot of proofs!

Some of the most common strategies are:

- a) proof by induction
- b) proof by contradiction
- c) proving the contrapositive

a) Proof by induction

Claim: Some statement $P(n)$
is true for every $n \in \mathbb{N}$.

Strategy:

- First prove $P(1)$
- Then prove that $P(n) \Rightarrow P(n+1)$,
for any $n \in \mathbb{N}$.

Proposition: $\sum_{k=1}^n k = \frac{n(n+1)}{2}$ for any $n \in \mathbb{N}$.

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Proof: • We have $\sum_{k=1}^1 k = 1$ and $\frac{1 \cdot (1+1)}{2} = 1$, so $P(1)$ is true.

• Assume $P(n)$ is true for some $n \in \mathbb{N}$. We have

$$\begin{aligned}\sum_{k=1}^{n+1} k &= \sum_{k=1}^n k + (n+1) = \frac{n(n+1)}{2} + (n+1) \\ &= \frac{(n+1)(n+2)}{2}\end{aligned}$$

so $P(n+1)$ is also true.



6) Proof by contradiction (reductio ad absurdum)

Strategy: To prove P , assume $(\text{not } P)$, and reach a contradiction.

Proposition: $\sqrt{2}$ is irrational

Proof: Assume the converse - that $\sqrt{2}$ is rational.

Let $p, q \in \mathbb{N}$ be such that $\sqrt{2} = p/q$.

We may assume that p and q have no common prime factors.

Then $2 = p^2/q^2 \iff p^2 = 2q^2$. Thus, p is even: $p = 2r$.

Therefore, $4r^2 = 2q^2 \iff q^2 = 2r^2$, so q is also even.

But p and q have no common prime factors!



c) Proving the contrapositive

Strategy: To prove $P \Rightarrow Q$, prove $(\text{not } Q) \Rightarrow (\text{not } P)$

Note: To prove $P \Leftrightarrow Q$, it is common to prove $P \Rightarrow Q$ and $(\text{not } P) \Rightarrow (\text{not } Q)$

Proposition: If $n \in \mathbb{N}$ is such that the num of its positive divisors is $n+1$, then n is prime

Proof: Let $n \in \mathbb{N}$ be non-prime. Then it is divisible by 1, n , and at least one more number $x \in \mathbb{N}$. Hence,

$$(\text{sum of positive divisors}) \geq 1 + n + x > n + 1.$$



QUESTIONS ?
COMMENTS ?

