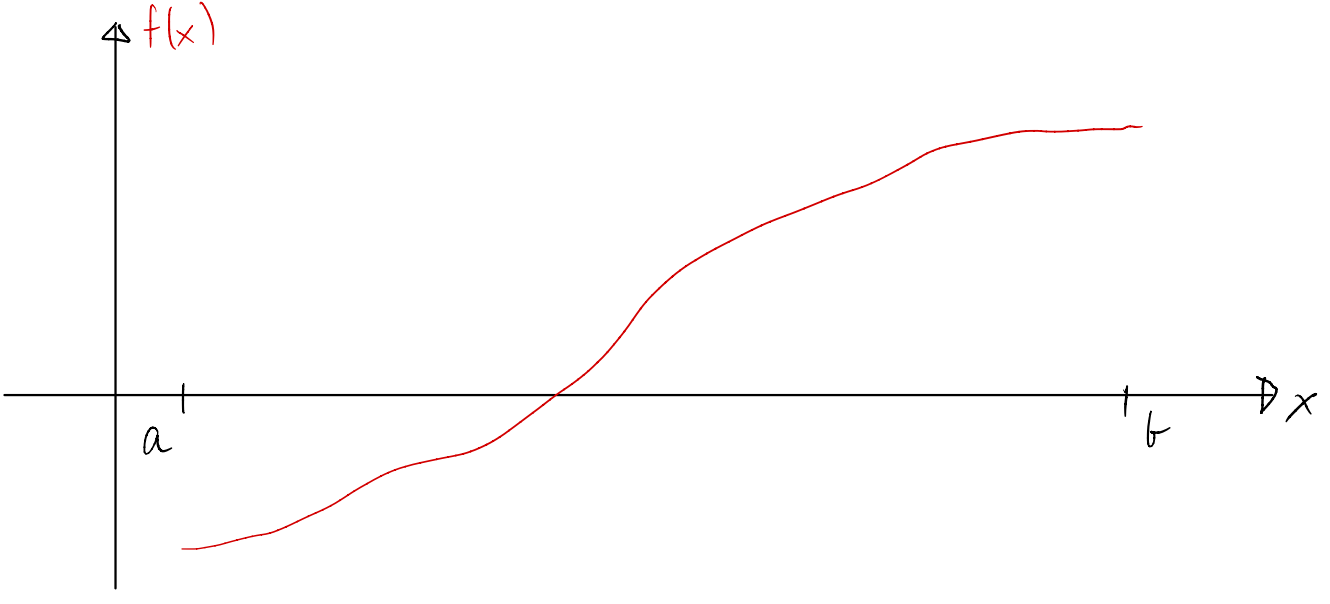
FOUR THEOREMS FROM CALCULUS

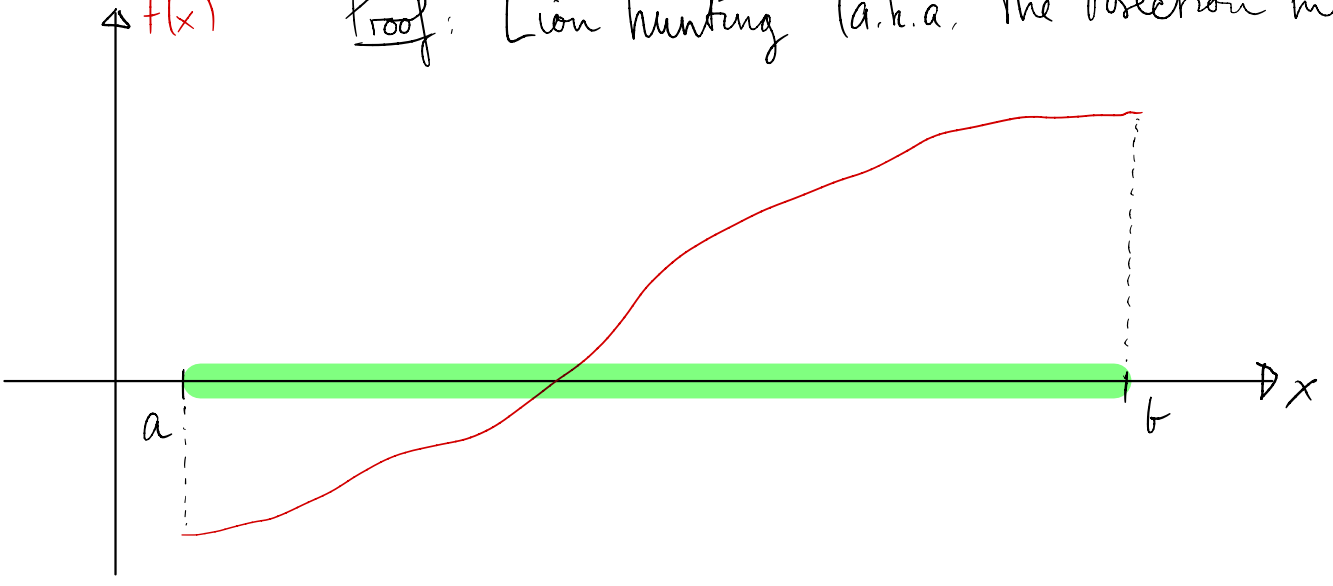
The intermediate value theorem

If $f : [a, b] \to \mathbb{R}$ is continuous and satisfies $f(a) > 0$, $f(b) < 0$, then there is some $c \in [a, b]$ where $f(c) = 0$.



The intermediate value theorem

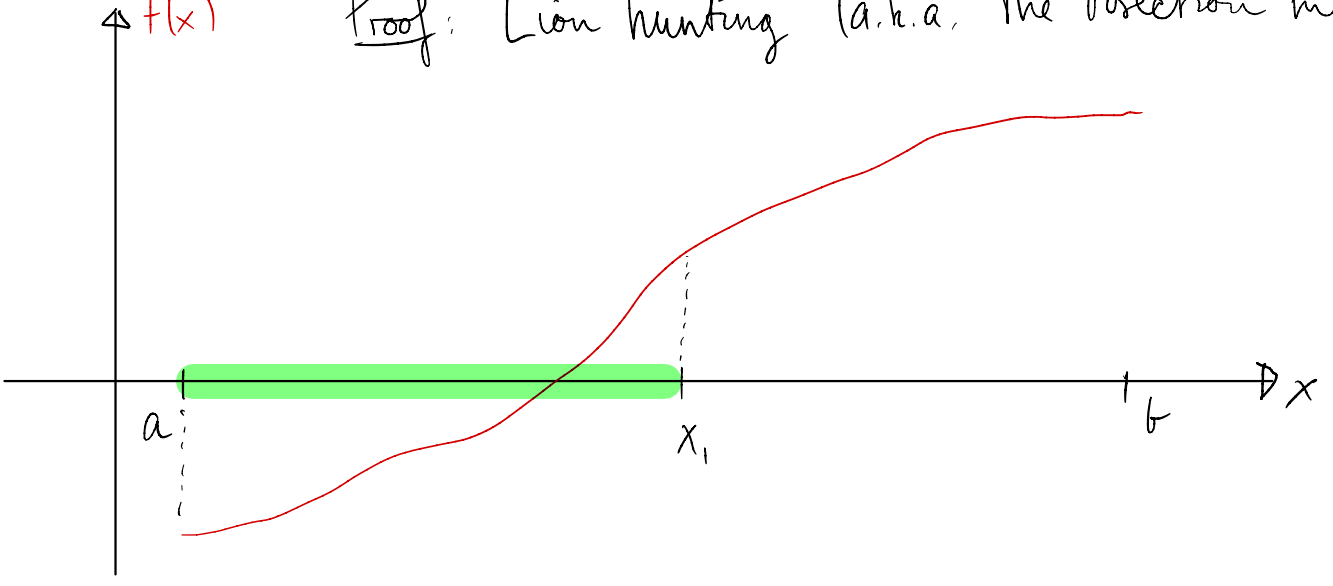
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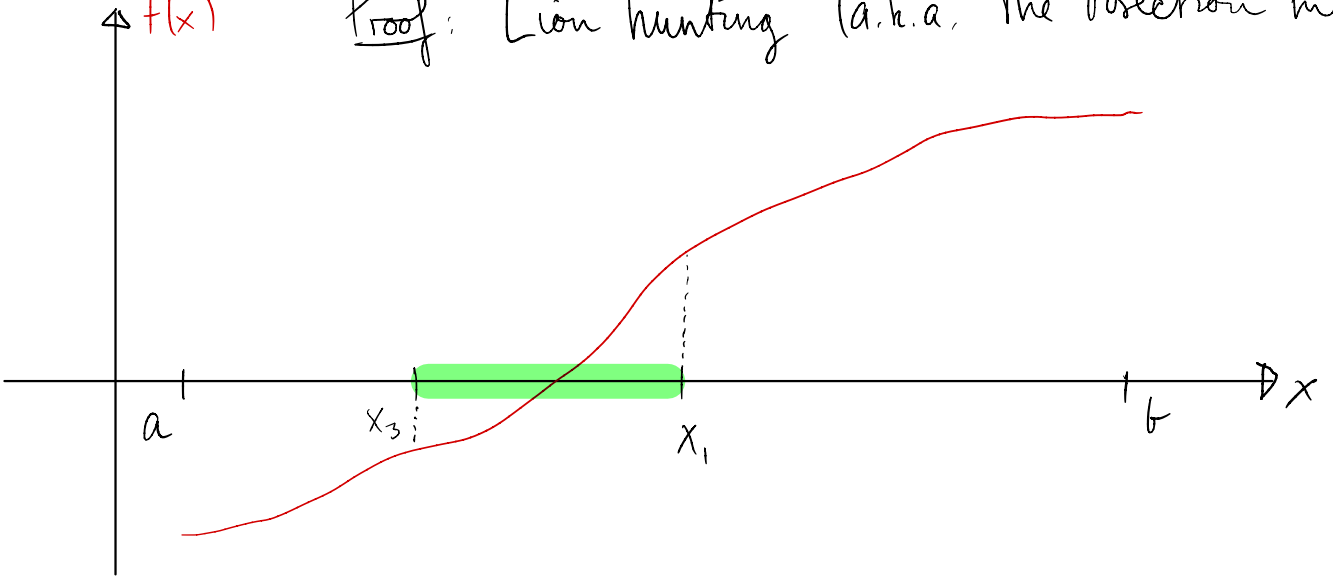
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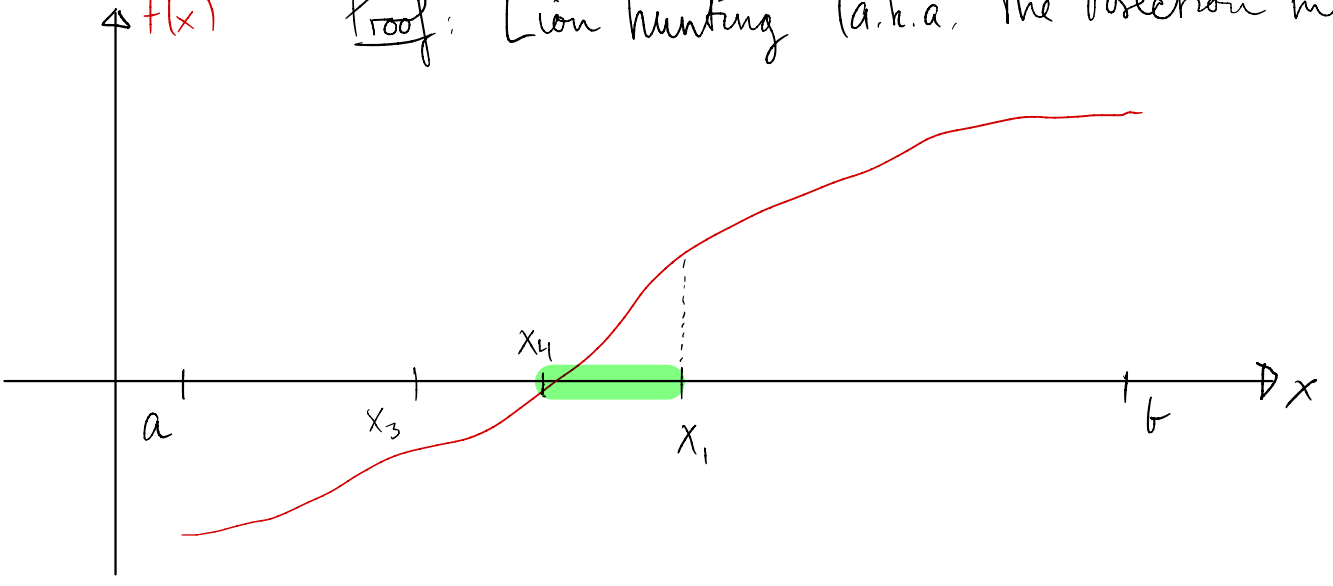
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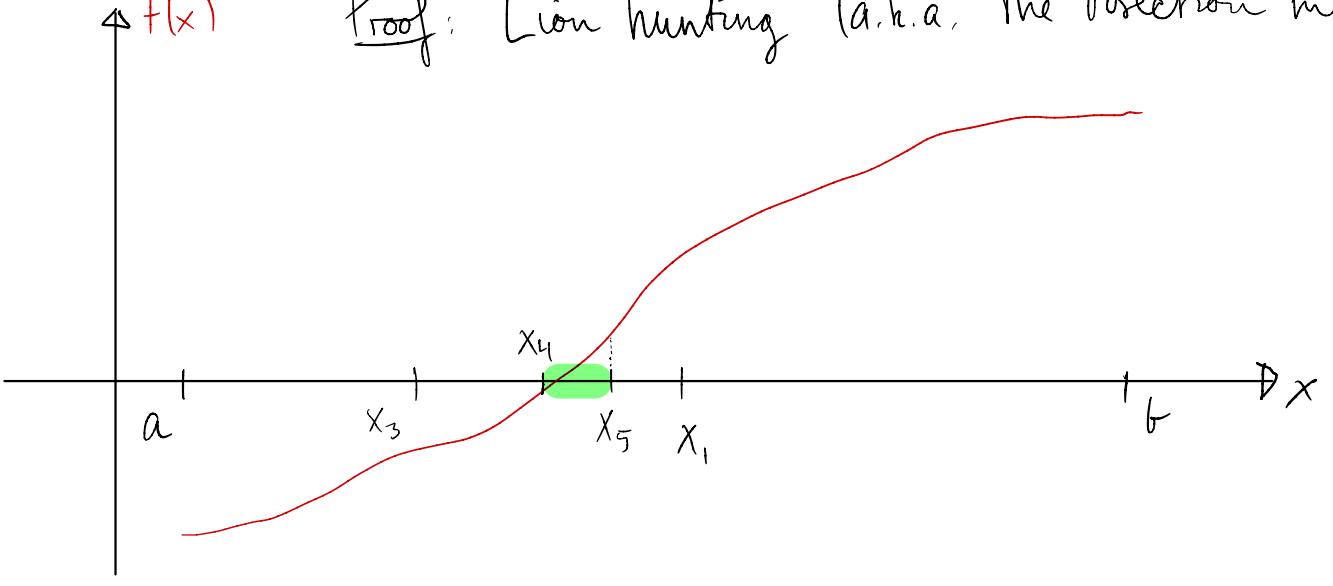
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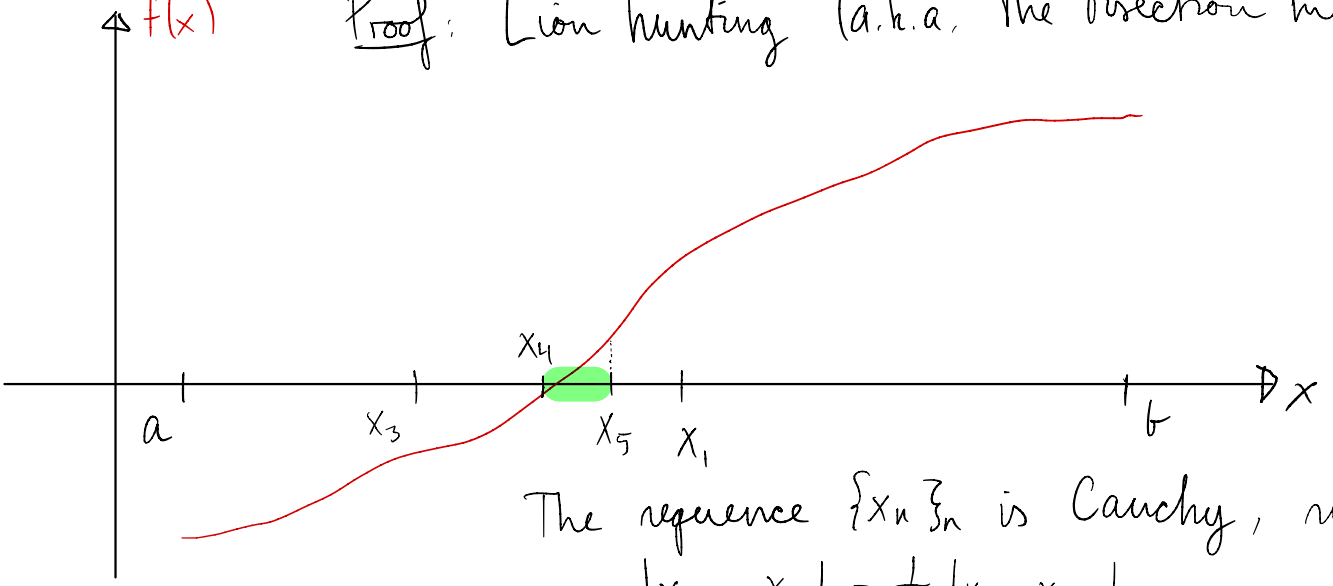
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The sequence $\{x_n\}_n$ is Cauchy, since

$$|x_{n+1} - x_n| = \frac{1}{2} |x_n - x_{n-1}|$$

■

The Bolzano - Weierstrass Theorem

If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence and $n(1) < n(2) < n(3) < \dots$ are natural numbers, then $\{x_{n(k)}\}_{k\in\mathbb{N}}$ is a subsequence of $\{x_n\}_{n}$

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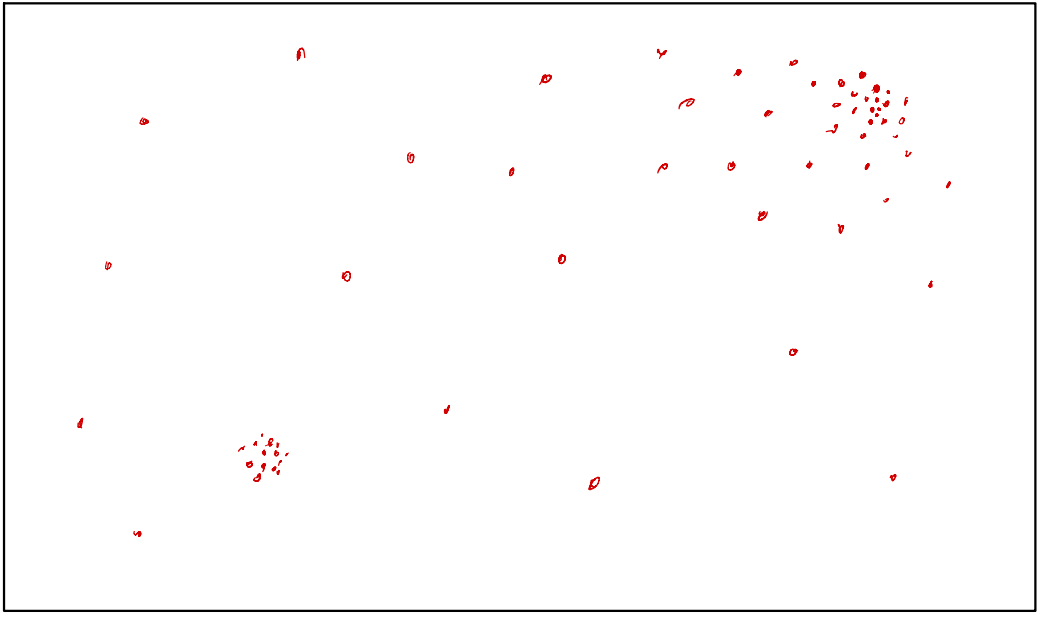
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A sequence $\{x_n\}_n$ in $\mathbb{R}^m$ is <u>bounded</u> if there is some $M > 0$ such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

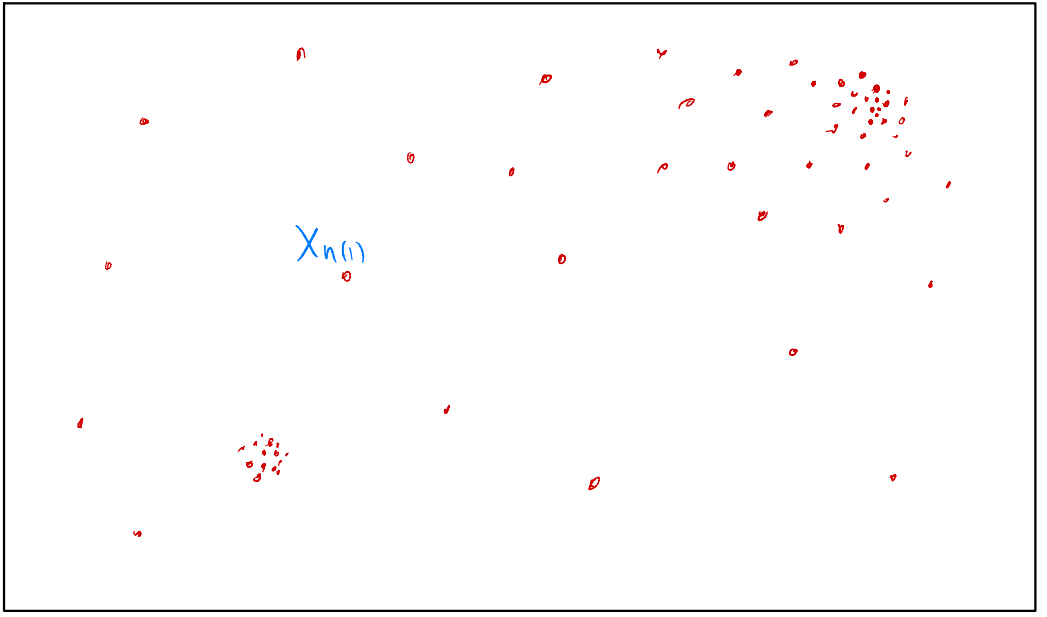
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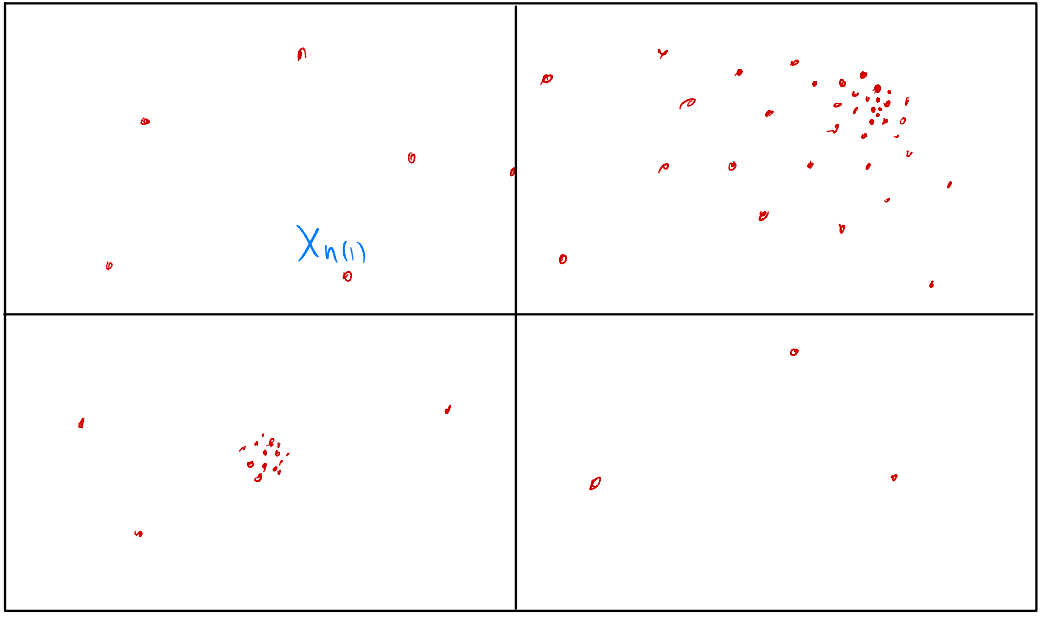
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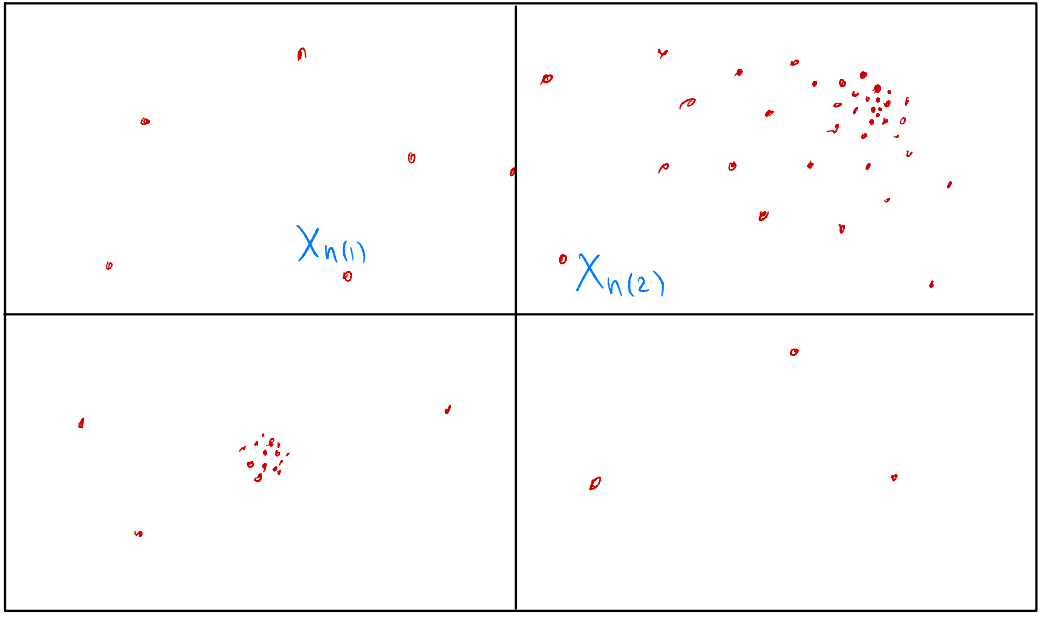
At least one of these subsets contains infinitely many objects



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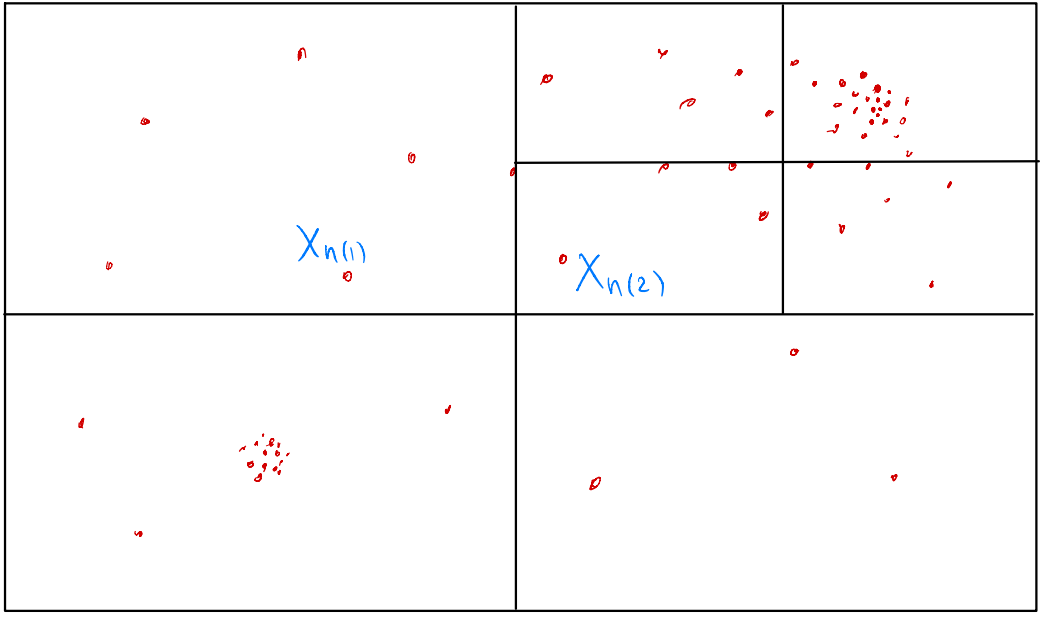
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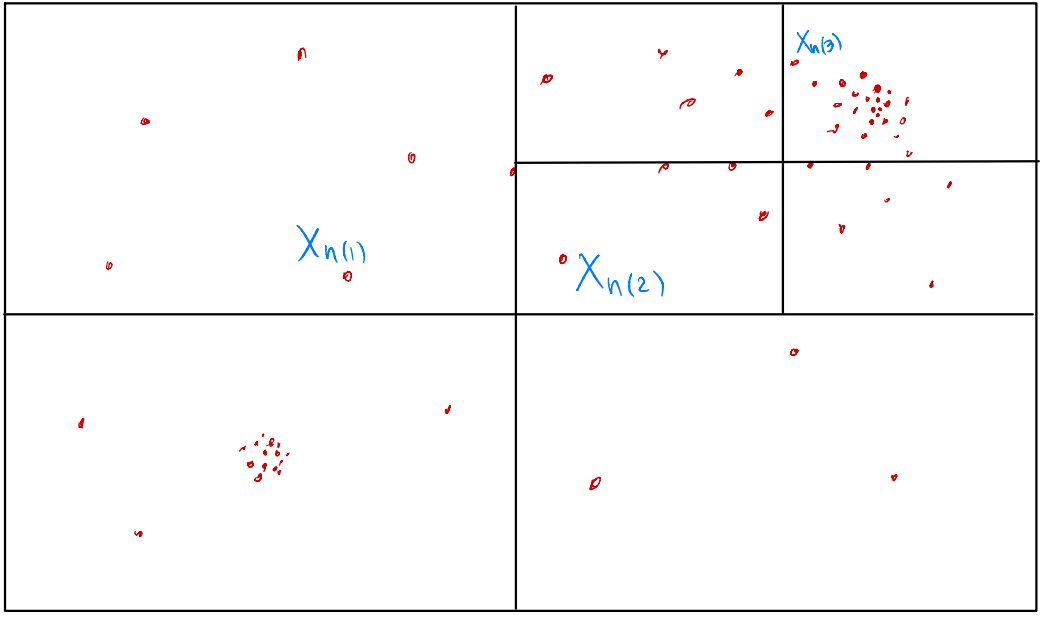
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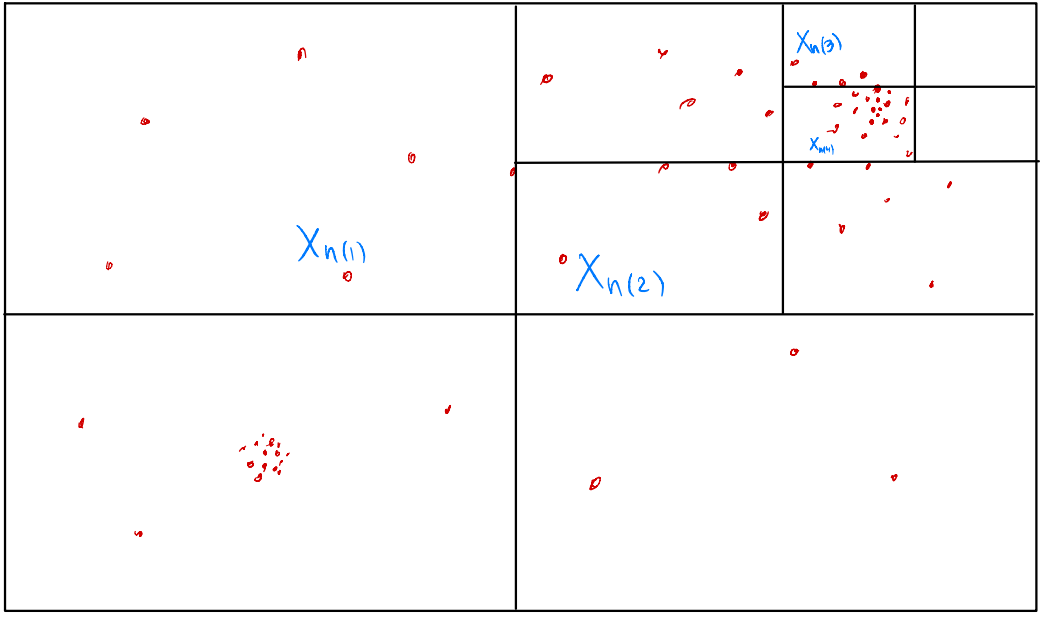
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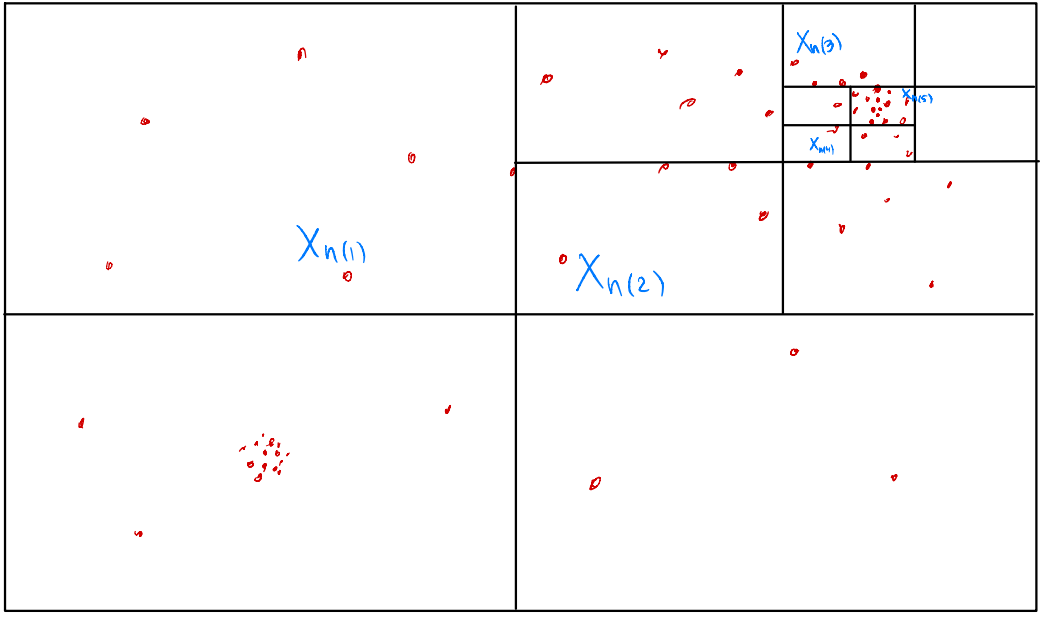
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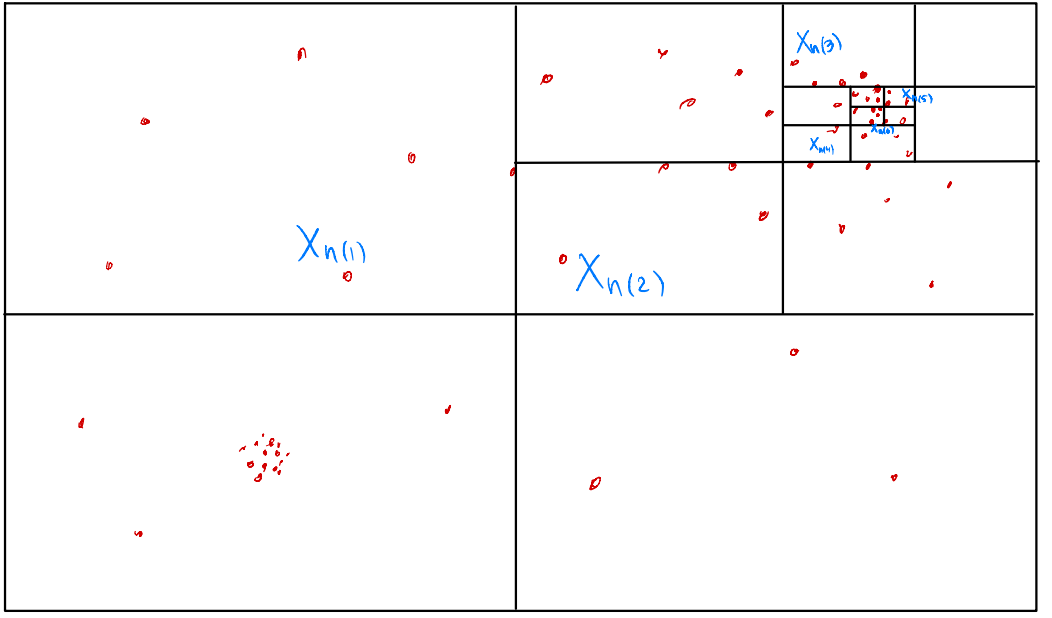
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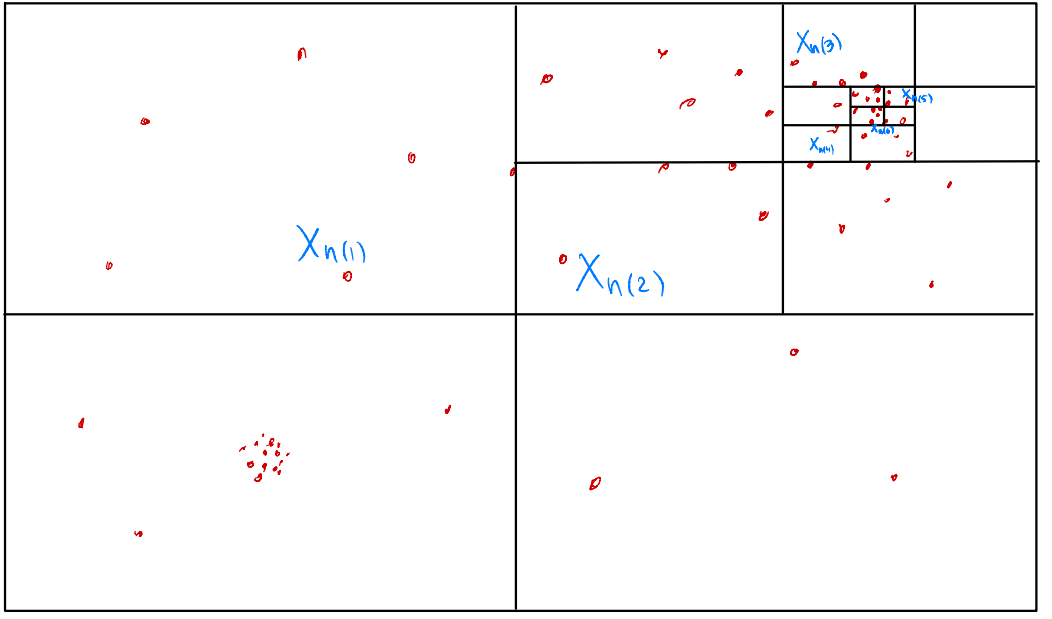


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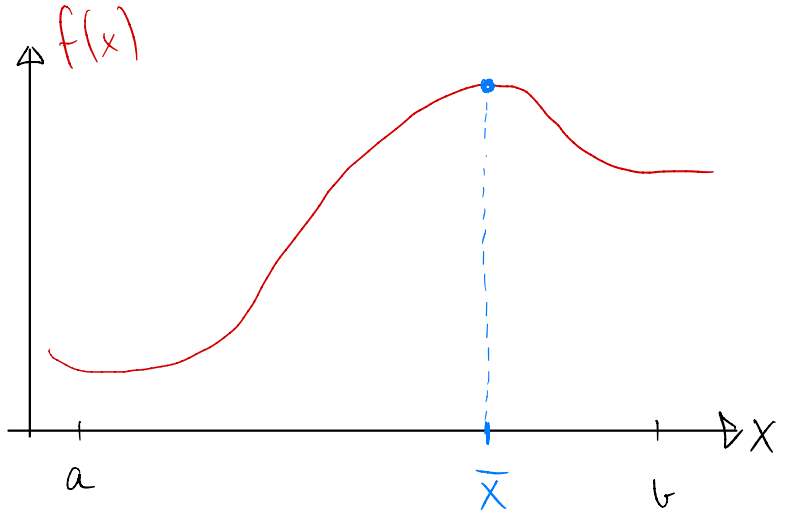
The sequence is "forced" into a smaller and smaller cage

$\Rightarrow$ Cauchy!



The extreme value theorem

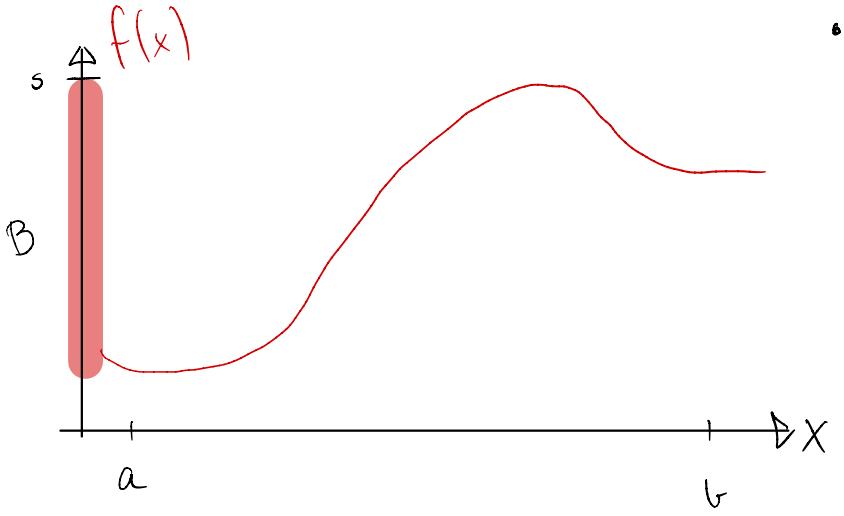
If $f:[a,b] \to \mathbb{R}$ is continuous then it has a maximum.



$$f(\overline{x}) \geq f(x) \quad \forall\ x \in [a,b]$$

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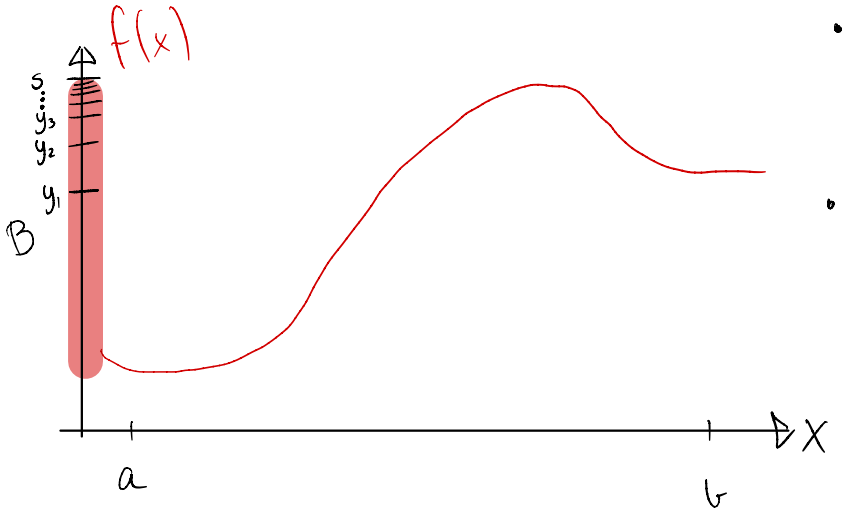
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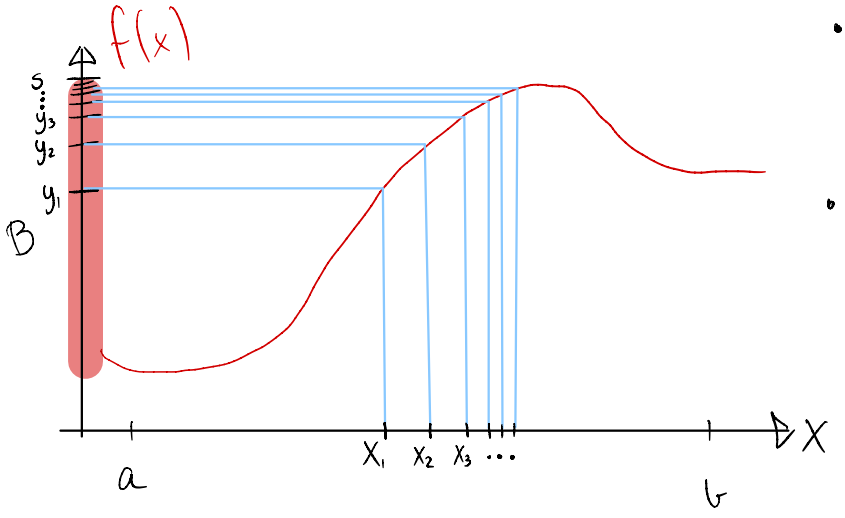
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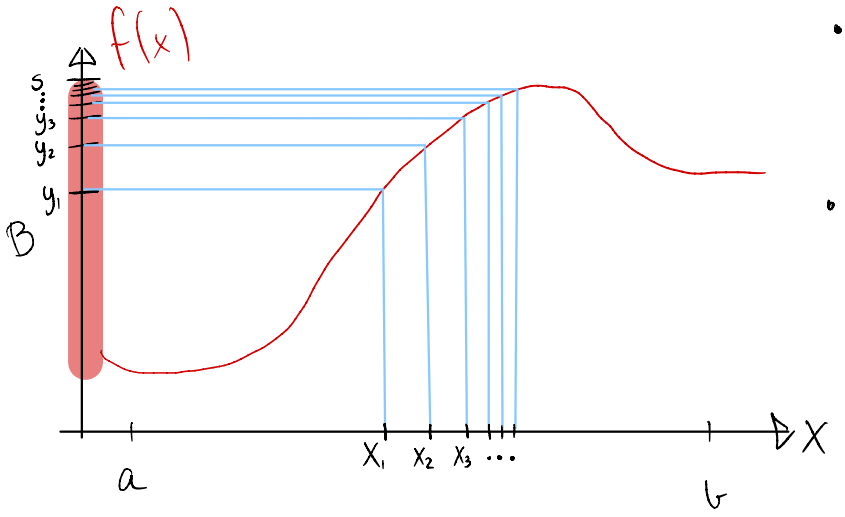
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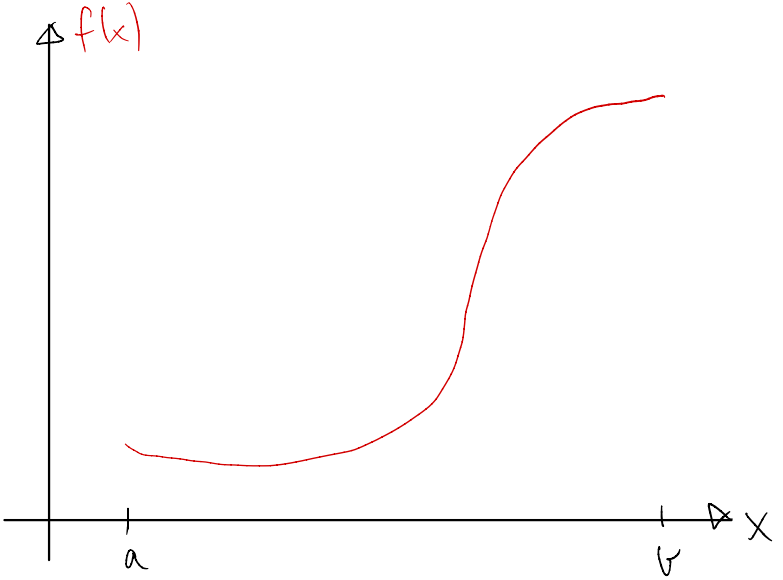
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- Let $\{x_n\}_n$ such that $f(x_n)=y_n$
- Use continuity to conclude $\{x_n\}_n$ is Cauchy
- Then $x_n \xrightarrow[n\to\infty]{} \bar{x}$, and $f(\bar{x})=f(\lim_n x_n)=\lim_n f(x_n)=\lim_n y_n = s$

■

The mean value theorem

If $f: [a,b] \to \mathbb{R}$ is continuously differentiable, then there is some $\bar{x} \in [a,b]$ where

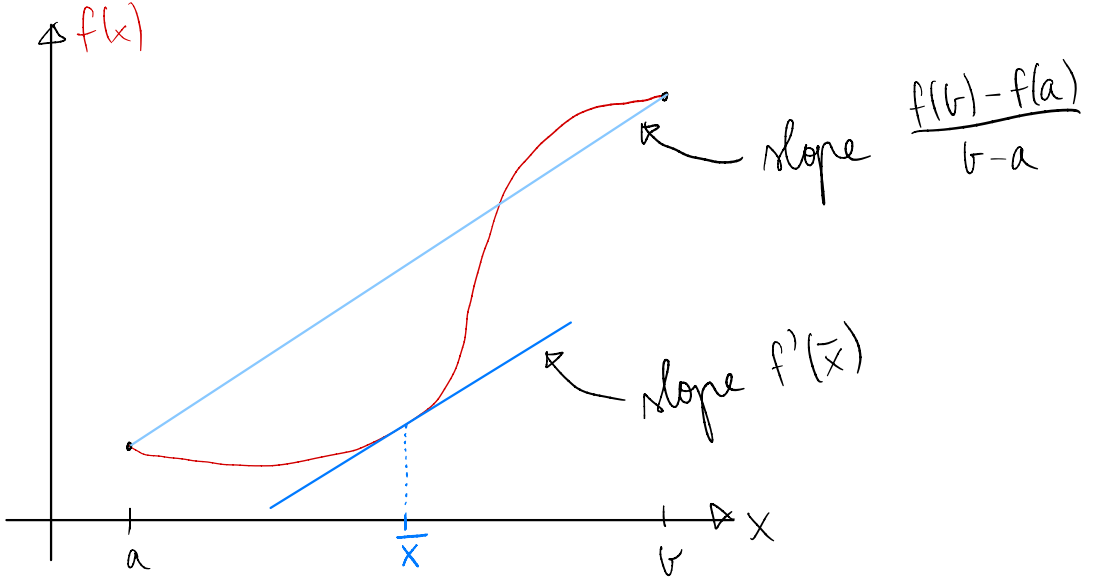
$$f'(\bar{x}) = \frac{f(b) - f(a)}{b - a}$$



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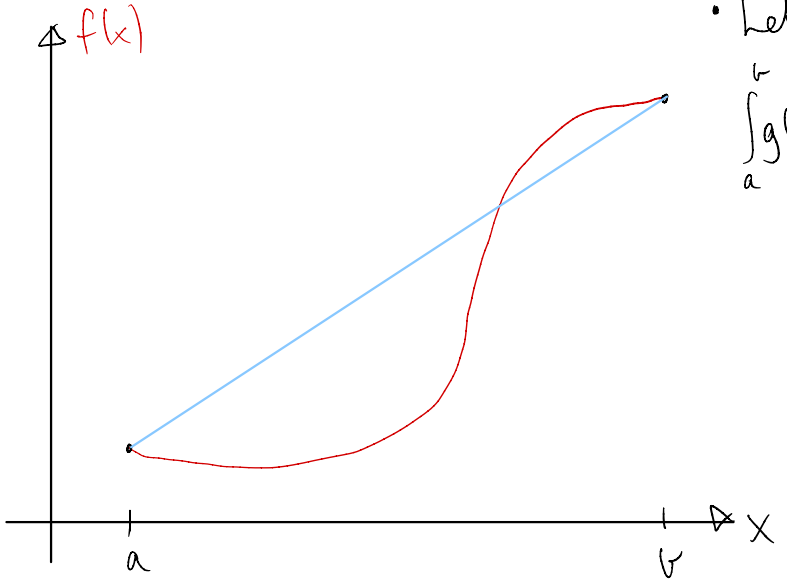
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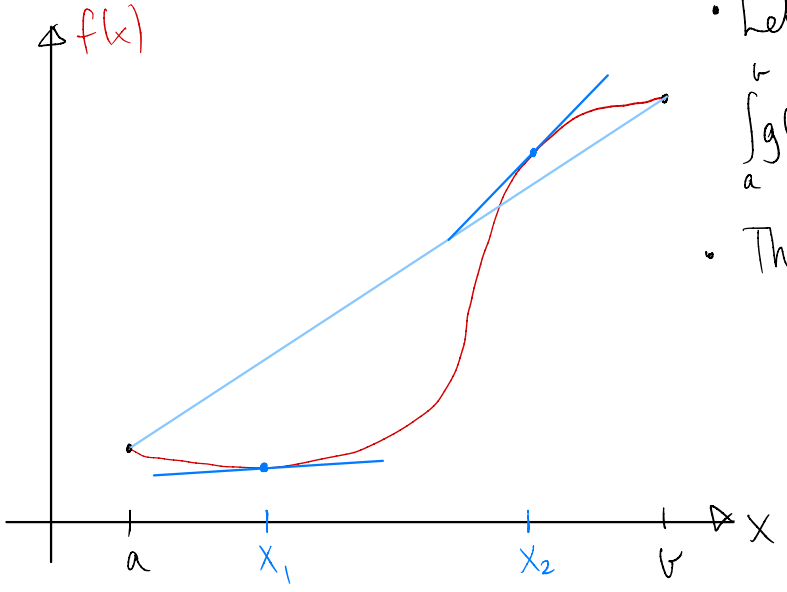
- Let $g(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a}$. Then

$$\int_a^b g(x)\,dx = f(b)-f(a)-(b-a)\frac{f(b)-f(a)}{b-a} = 0.$$

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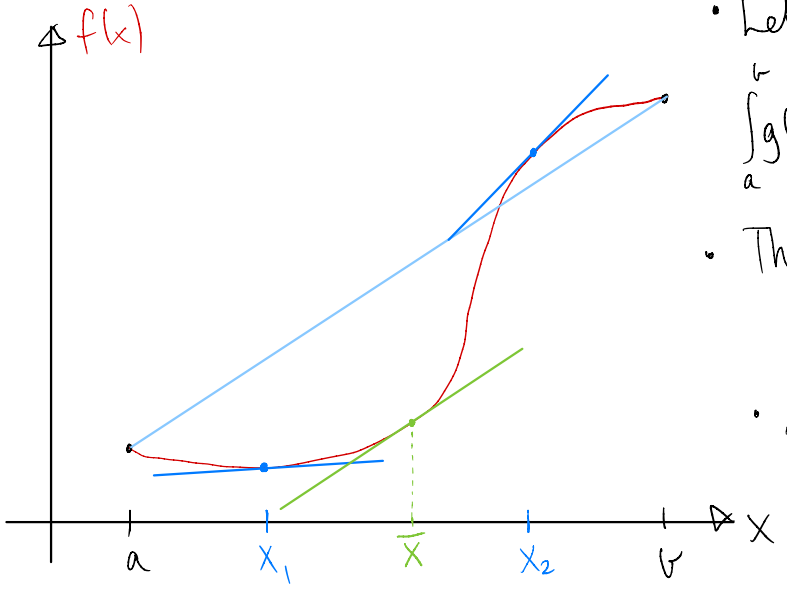
- Thus, $\exists\ x_1, x_2 \in [a,b]$ where

$$g(x_1)<0, \quad g(x_2)>0.$$

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- Thus, $\exists\ x_1, x_2 \in [a,b]$ where

$$g(x_1)<0, \quad g(x_2)>0.$$

- Apply intermediate value theorem:

$$\exists\ \bar{x}: \quad g(\bar{x}) = 0$$



Note

In all four theorems, continuity and completeness play central roles.

QUESTIONS ?

COMMENTS ?