CARDINALITY

If A is a set, then $\#A$ = (number of elements in A) is the cardinality of A.

Example: If $A = \{$  $\}$ then $\#A = 4$. To compute $\#A$ we count: $\{1, 2, 3, 4\}$.

There exists a surjective function $f: \{1,2,3,4\} \twoheadrightarrow A$.

How can we "count" infinitely large sets?

A set B is <u>countable</u> if there exists a surjective function $f : \mathbb{N} \twoheadrightarrow B$

Thus, we can write B as a list: $B = \{f(1), f(2), f(3), \ldots\}$

<u>Examples</u>:

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

$\{1, 2, 3\} = \{1, 2, 3, 1, 2, 3, \ldots\}$

Any finite set

$2\mathbb{N} = \{2, 4, 6, 8, \ldots\}$

$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$

Intuitively, a set is countable if it contains "as many" elements as $\mathbb{N}$ does (or fewer).

If B and C are countable, then so is $B \times C$

Recall: $B \times C = \{(b, c) : b \in B, c \in C\}$

Proof: Let $f : \mathbb{N} \to B$ and $g : \mathbb{N} \to C$ be surjective, so that $B = \{f(1), f(2), f(3), \dots\}$ and $C = \{g(1), g(2), g(3), \dots\}$.

	$f(1)$	$f(2)$	$f(3)$	$f(4)$
$g(1)$	$(f(1), g(1))$ ①	$(f(2), g(1))$ ③	$(f(3), g(1))$ ⑥	$(f(4), g(1))$ ⑩
$g(2)$	$(f(1), g(2))$ ②	$(f(2), g(2))$ ⑤	$(f(3), g(2))$ ⑨	
$g(3)$	$(f(1), g(3))$ ④	$(f(2), g(3))$ ⑧	$(f(3), g(3))$	
$g(4)$	$(f(1), g(4))$ ⑦			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	



Corollary: If $B_1, \dots, B_n$ are countable, then so is $B_1 \times \cdots \times B_n$.

$$B_1 \times (B_2 \times \cdots \times B_n)$$

Corollary: $\mathbb{Q}$ is countable

Proof: We have $\mathbb{Q} = \{p/q : p \in \mathbb{Z},\ q \in \mathbb{N}\}$. Therefore,

$f : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$, $f(p,q) = p/q$ is surjective.

The set $\mathbb{Z} \times \mathbb{N}$ is countable, so there exists a surjective $g : \mathbb{N} \to \mathbb{Z} \times \mathbb{N}$. Hence,

$h : \mathbb{N} \to \mathbb{Q}$, $h = f \circ g$

is surjective.



If $B_1, \ldots, B_n$ are countable then so is $B_1 \cup \cdots \cup B_n$.

If $B_1, B_2, \ldots$ are countable then so is $\bigcup_{k=1}^{\infty} B_k$.

Proof: Exercise! (Use the same idea as in Cartesian products.)

$\mathbb{R}$ is not countable

Proof: Assume the converse: $\exists$ surjective $f : \mathbb{N} \to \mathbb{R}$.

Let $a_0^n . a_1^n a_2^n a_3^n \cdots$ be the decimal expansion of $f(n)$ (so $a_0^n \in \mathbb{Z}$ and $a_1^n, a_2^n, \ldots \in \{0, 1, \ldots, 9\}$):

$f(1) = a_0^1 . a_1^1 a_2^1 a_3^1 a_4^1 \cdots$

$f(2) = a_0^2 . a_1^2 a_2^2 a_3^2 a_4^2 \cdots$

$f(3) = a_0^3 . a_1^3 a_2^3 a_3^3 a_4^3 \cdots$

$f(4) = a_0^4 . a_1^4 a_2^4 a_3^4 a_4^4 \cdots$

$\vdots$

Let $b^n = \begin{cases} 1 & \text{if } a_n^n \neq 1 \\ 2 & \text{if } a_n^n = 1 \end{cases}$

Then $x = 0 . b^1 b^2 b^3 \cdots$ does not appear in the list!

↯

∎

QUESTIONS?

COMMENTS?