VECTOR SPACES AND NORMS

A <u>vector space</u> (over $\mathbb{R}$) is a set V equipped with two operations, addition and multiplication by scalars, such that

i) $u+v = v+u \quad \forall\ u,v \in V$

ii) $(u+v)+w = u+(v+w) \quad \forall\ u,v,w \in V$

iii) $1 \cdot u = u \quad \forall\ u \in V$

iv) there is some $0 \in V$ such that $u+0=u \quad \forall\ u \in V$

v) for all $u \in V$ there is some $-u \in V$ such that $u+(-u)=0$

vi) $\alpha(\beta u) = (\alpha\beta)u \quad \forall\ u \in V,\ \alpha,\beta \in \mathbb{R}$

vii) $\alpha(u+v) = \alpha u + \alpha v \quad \forall\ u,v \in V,\ \alpha \in \mathbb{R}$

viii) $1u = u \quad \forall\ u \in V$

iv) $(\alpha+\beta)u = \alpha u + \beta u \quad \forall\ u \in V,\ \alpha,\beta \in \mathbb{R}$

Examples:

$\mathbb{R}$

$\mathbb{R}^n$

$\mathbb{P}_n = \{\text{all polynomials of order at most } n\}$

$\mathbb{P} = \{\text{all polynomials}\}$

A norm on a vector space V is a function $\|\cdot\| : V \to [0, \infty)$ satisfying:

Positivity: $\|u\| \geq 0 \quad \forall\ u \in V$, and $\|u\| = 0$ only if $u = 0$

Homogeneity: $\|\alpha u\| = |\alpha| \, \|u\| \quad \forall\ u \in V,\ \alpha \in \mathbb{R}$

Triangle inequality: $\|u+v\| \leq \|u\| + \|v\| \quad \forall\ u, v \in V$

A vector space with a norm is a normed vector space

We think of $\|u\|$ = length of u

$\|u-v\|$ = distance between u and v

Example:

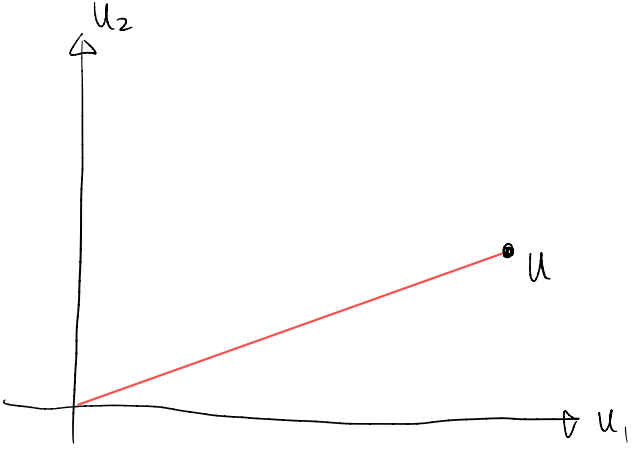
$V = \mathbb{R}$ and $\|u\| = |u|$ (the absolute value).

This is the canonical norm on $\mathbb{R}$

Example:

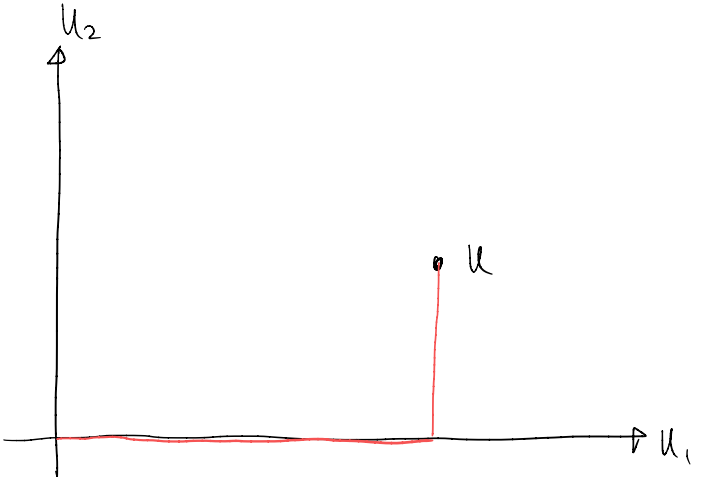
$V = \mathbb{R}^n$ and $\|u\| = \sqrt{u_1^2 + \dots + u_n^2}$ (where $u = (u_1, \dots, u_n)$)

This is the Euclidean or canonical norm on $\mathbb{R}^n$



Example:

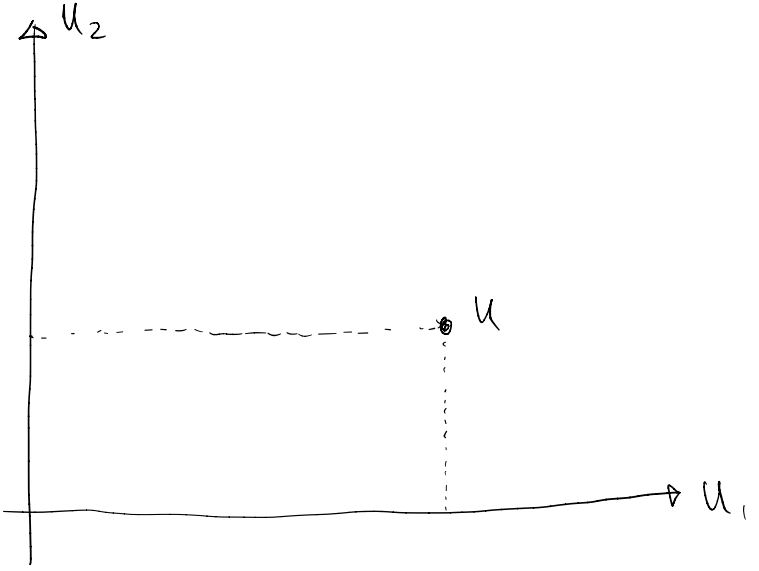
$V = \mathbb{R}^n$ and $\|u\| = |u_1| + \cdots + |u_n|$. This is the l_1 norm (or taxicab norm, or Manhattan norm)



Example:

$V = \mathbb{R}^n$ and $\|u\| = \max\{|u_1|, \ldots, |u_n|\}$.

This is the l_∞-norm.



QUESTIONS ?

COMMENTS ?