CONVERGENCE AND CONTINUITY

Let V be a normed vector space, let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in V and let $x \in V$. We say that $\{x_n\}_n$ converges to x if for all $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$\|x_n - x\| < \varepsilon \qquad \forall \; n \geq N.$$

$$x_1 \quad x_2 \quad x_3 \quad \cdots\cdots \quad \underbrace{x_N \quad x_{N+1} \quad \cdots}_{}$$

all elements closer to x than ε

The following mean the same:

- $\{x_n\}_n$ converges to x
- $\lim_{n\to\infty} x_n = x$
- $x_n \longrightarrow x$ as $n \to \infty$
- $x_n \xrightarrow[n\to\infty]{} x$

The following is meaningless:

- x_n converges to x
- $\lim x_n = x$

Example: $V = \mathbb{R}$, $\|\cdot\| = |\cdot|$, $x_n = \frac{1}{n}$, for $n \in \mathbb{N}$.

We guess that $x_n \longrightarrow 0$ as $n \to \infty$. Let $\varepsilon > 0$.

We have $\|x_n - 0\| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} \overset{!}{<} \varepsilon \iff \frac{1}{\varepsilon} \overset{!}{<} n$.

Thus, we choose N to be any integer $> \frac{1}{\varepsilon}$. Then

$$\|x_n - 0\| < \varepsilon \qquad \text{when } n \geq N.$$

Example: $V = \mathbb{R}^3$ with the Euclidean norm, $x_n = \begin{pmatrix} 1/n \\ -5/n \\ 1/n^2 \end{pmatrix}$, $n \in \mathbb{N}$

Then $x_n \longrightarrow 0$ as $n \to \infty$.

Indeed, $\|x_n - 0\| = \|x_n\| \leq \left|\frac{1}{n}\right| + \left|-\frac{5}{n}\right| + \left|\frac{1}{n^2}\right| \longrightarrow 0$ as $n \to \infty$.

Proposition: If $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then

$$x_n + y_n \longrightarrow x+y \quad \text{as } n \to \infty.$$

Proof: Let $\varepsilon > 0$.

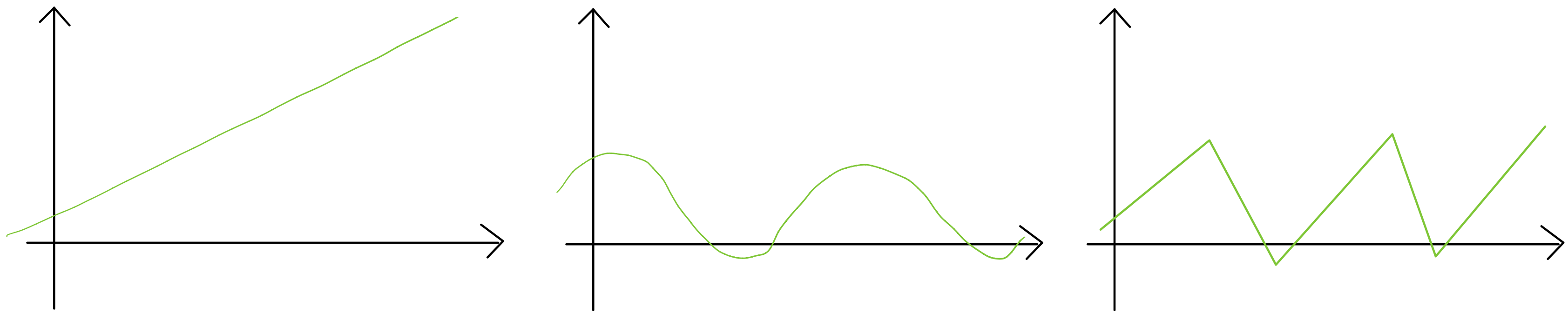
$$\begin{array}{cccccc} x_1 & x_2 & x_3 & \cdots\cdots & x_N & x_{N+1} \quad \cdots \\ y_1 & y_2 & y_3 & \cdots\cdots\cdots\cdots & y_M & y_{M+1} \quad \cdots \end{array}$$

If $\|x_n - x\| < \varepsilon$ for $n \geq N$, and $\|y_n - y\| < \varepsilon$ for $n \geq M$, then $\|x_n - x\| + \|y_n - y\| < 2\varepsilon$ for $n \geq \max(N, M)$. Hence,

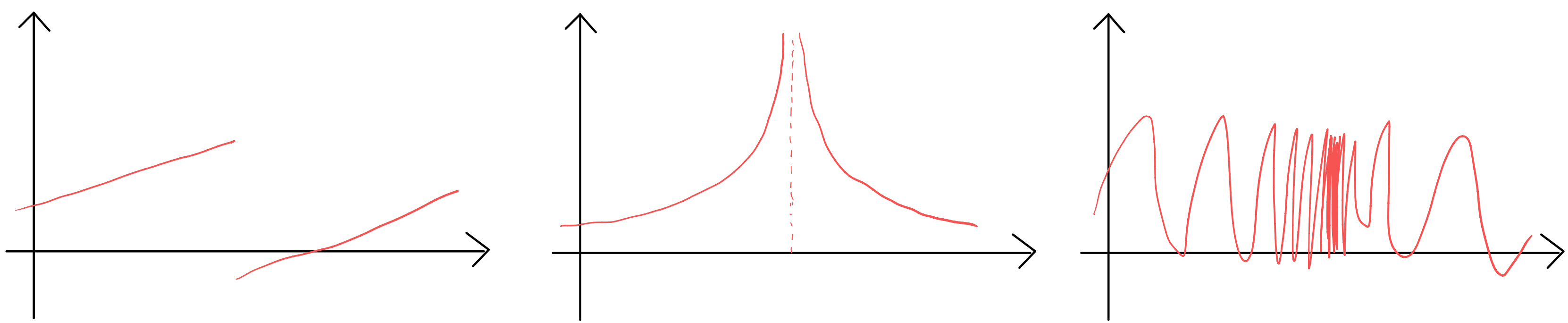
$$\|(x_n + y_n) - (x+y)\| = \|(x_n - x) + (y_n - y)\| \leq \|x_n - x\| + \|y_n - y\| < 2\varepsilon.$$

$\blacksquare$

Continuity



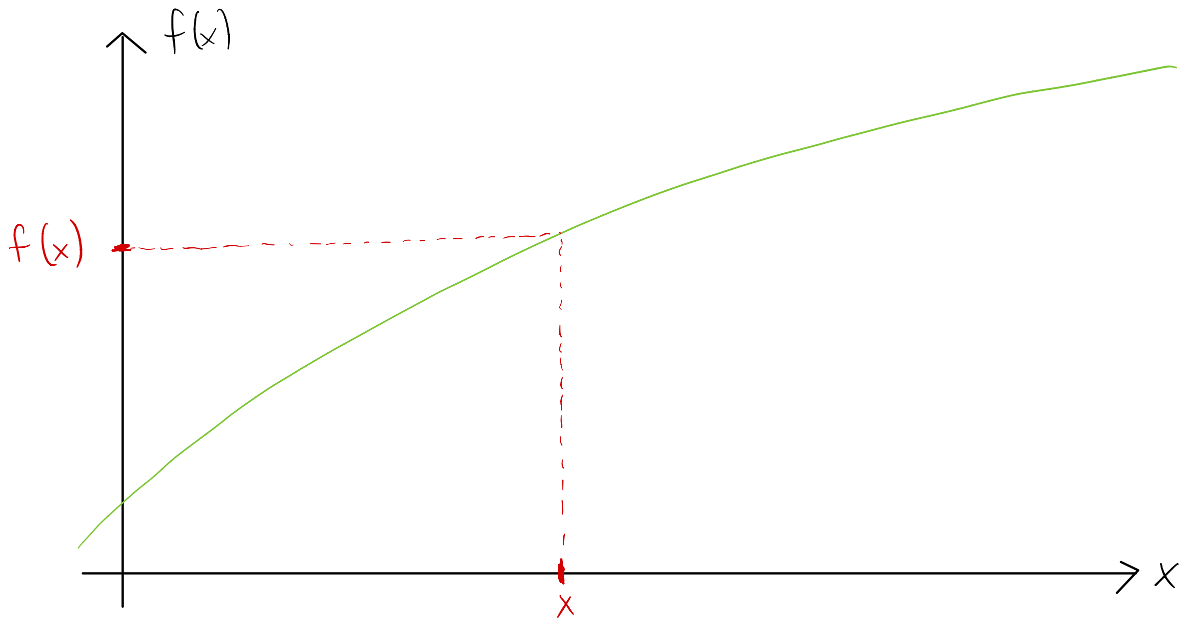
Continuous



Discontinuous

A function $f: \mathbb{R} \to \mathbb{R}$ is <u>continuous</u> at x if for all $\varepsilon > 0$ there is some $\delta > 0$ such that

$|f(x) - f(y)| < \varepsilon$ when y is such that $|x-y| < \delta$



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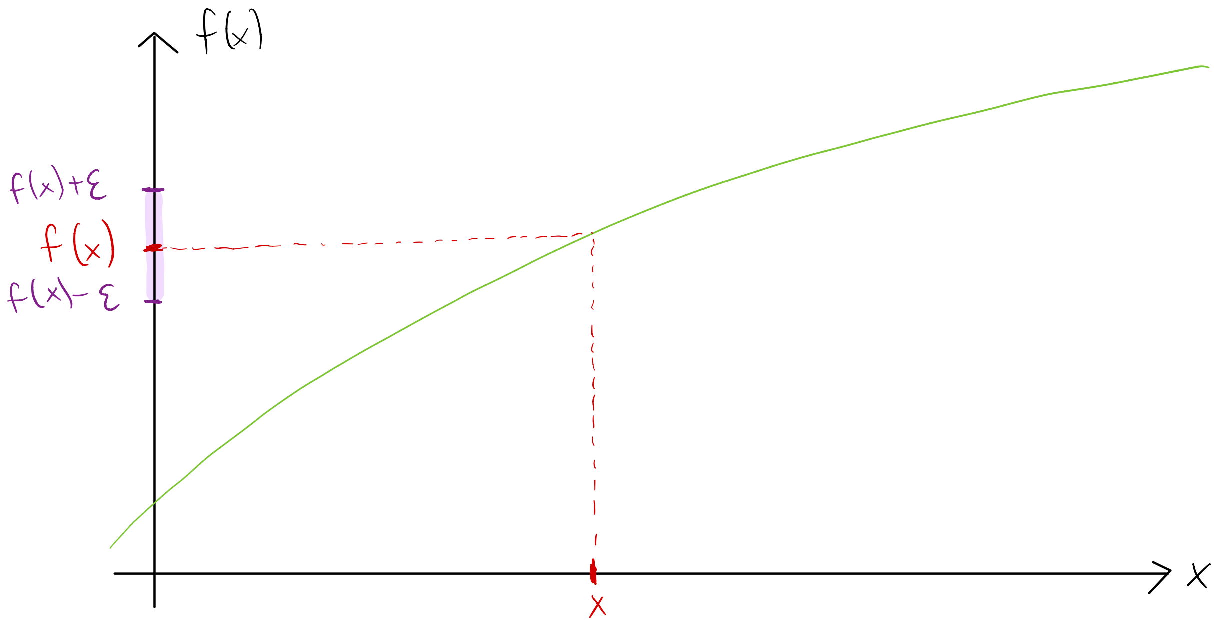

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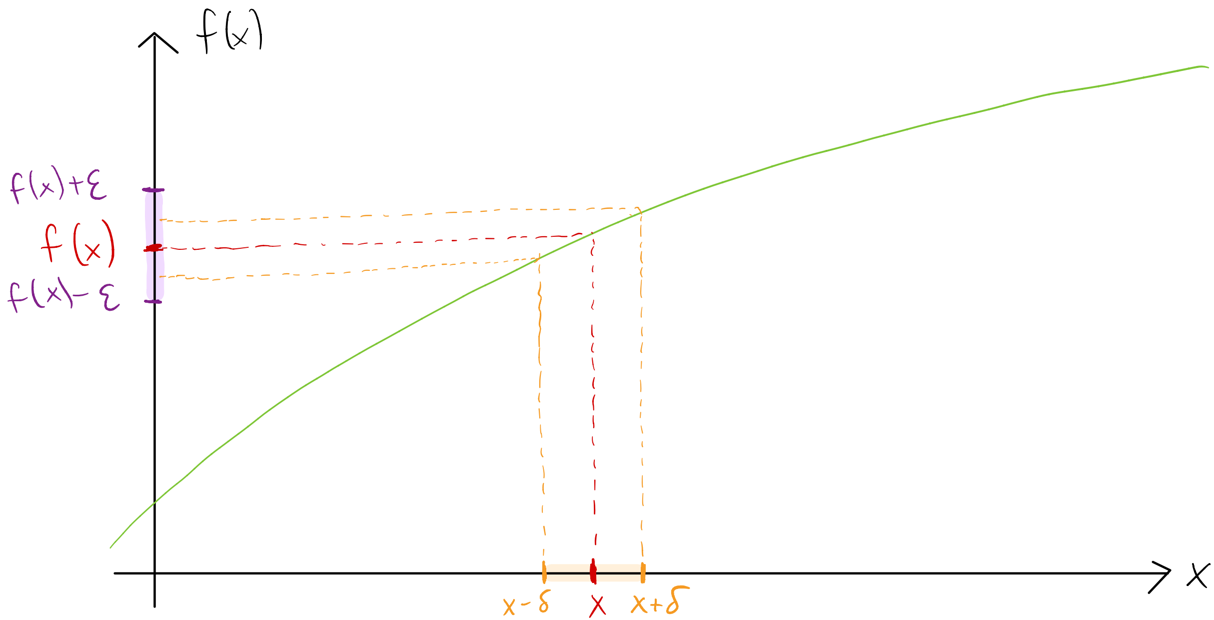
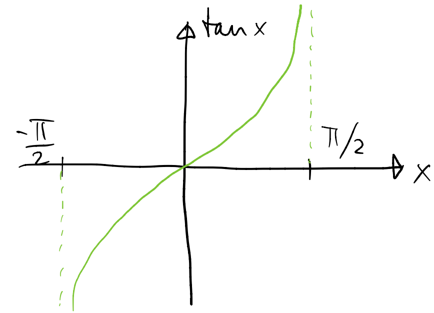


A function $f : B \to \mathbb{R}$ (where $B \subseteq \mathbb{R}$) is continuous at $x \in B$ if for every $\varepsilon > 0$ there is some $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ when $y \in B$ is such that $|x - y| < \delta$.

<u>Example</u>: tan is continuous in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, but not outside.



A function $f : B \to \mathbb{R}$ is <u>everywhere continuous</u> (or simply <u>continuous</u>) if it is continuous at every $x \in B$

A function $f : B \to \mathbb{R}^n$ (for some $B \subset \mathbb{R}^m$) is <u>continuous</u> at $x \in B$ if for every $\varepsilon > 0$ there is some $\delta > 0$ s.t.

$$\|f(x) - f(y)\| < \varepsilon \quad \text{when } y \in B \text{ is such that } \|x - y\| < \delta$$

Here, $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^m$ and $\mathbb{R}^n$.

QUESTIONS ?

COMMENTS ?