MIN, MAX, INF AND SUP

Let $B \subseteq \mathbb{R}$. By $\max B$ we mean some $x \in B$ satisfying

$x \geq y$ for all $y \in B$

Let $B \subseteq \mathbb{R}$. By $\min B$ we mean some $x \in B$ satisfying

$x \leq y$ for all $y \in B$

Example:

If $B = \{3, 8, 2, -4, 1\}$ then $\min B = -4$ and $\max B = 8$

If $B = [3, 7]$ then $\min B = 3$ and $\max B = 7$

Every nonempty, finite set $(\#B > 0$ and $\#B < \infty)$ has a minimum and a maximum.

Proof: Write $B = \{b_1, b_2, \ldots, b_n\}$. The following computes $M = \max B$:

$M \leftarrow b_1$
for $k = 2, \ldots, n$:
    if $b_k > M$:
        $M \leftarrow b_k$
    end
end



But min/max might not exist if B is infinitely large!

(i) B might be unbounded: $B = \mathbb{N} = \{1, 2, 3, \ldots\}$ has a min, but no max.

(ii) B might have a "hole": $B = (0, 1]$ has a max, but no min.

We first take care of problem (i):

- Let $B \subseteq \mathbb{R}$. M is an <u>upper bound for B</u> if $M \geq x$ for all $x \in B$.
- B is <u>upper bounded</u> if it has an upper bound.
- M is a <u>least upper bound</u> for B if all other upper bounds N satisfy $N \geq M$.

- Let $B \subseteq \mathbb{R}$. M is a <u>lower bound for B</u> if $M \leq x$ for all $x \in B$.
- B is <u>lower bounded</u> if it has a lower bound.
- M is a <u>greatest lower bound</u> for B if all other lower bounds N satisfy $N \leq M$.

Theorem (the completeness principle)

Let $B \subseteq \mathbb{R}$ be nonempty.

- If B is upper bounded then it has a least upper bound.
- If B is lower bounded then it has a greatest lower bound.

We denote the least upper bound $\sup B$ (supremum of B) and the greatest lower bound $\inf B$ (infimum of B).

Example:

$\inf \mathbb{N} = 1$

$\inf (0,1] = 0$

$\sup (0,1] = 1$

For an arbitrary $B \subseteq \mathbb{R}$ we write:

$$\sup B = \begin{cases} -\infty & \text{if } B \text{ is empty} \\ \infty & \text{if } B \text{ is not bounded from above} \\ \text{(least upper bound)} & \text{otherwise} \end{cases}$$

$$\inf B = \begin{cases} \infty & \text{if } B \text{ is empty} \\ -\infty & \text{if } B \text{ is not bounded from below} \\ \text{(greatest lower bound)} & \text{otherwise} \end{cases}$$

Theorem (the completeness principle)

Let $B \subseteq \mathbb{R}$ be nonempty.

- If B is upper bounded then it has a least upper bound.
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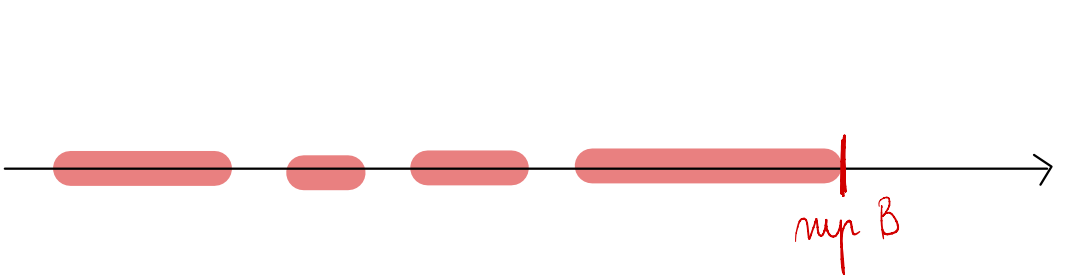
The completeness principle is essential because it "fills the holes" in $\mathbb{Q}$.

Example: If $B = \{x \in \mathbb{Q} : x^2 < 2\}$ then $\inf B = -\sqrt{2}$, $\sup B = \sqrt{2}$.

Maximizing sequences

Let $B \subseteq \mathbb{R}$ be nonempty and upper bounded. Then there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in B such that

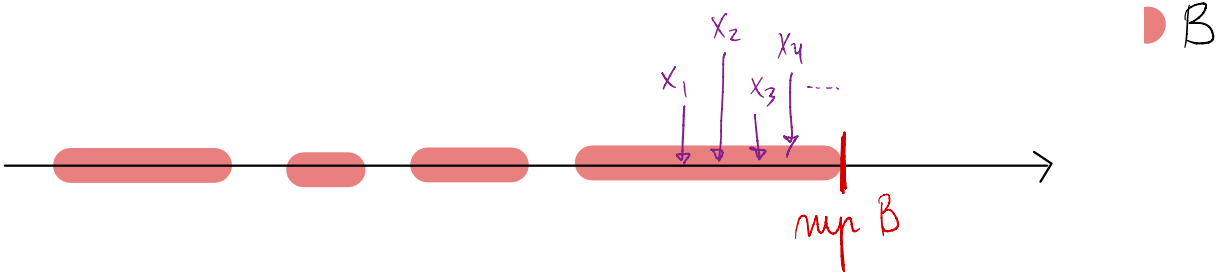
$$x_n \xrightarrow[n \to \infty]{} \sup B.$$

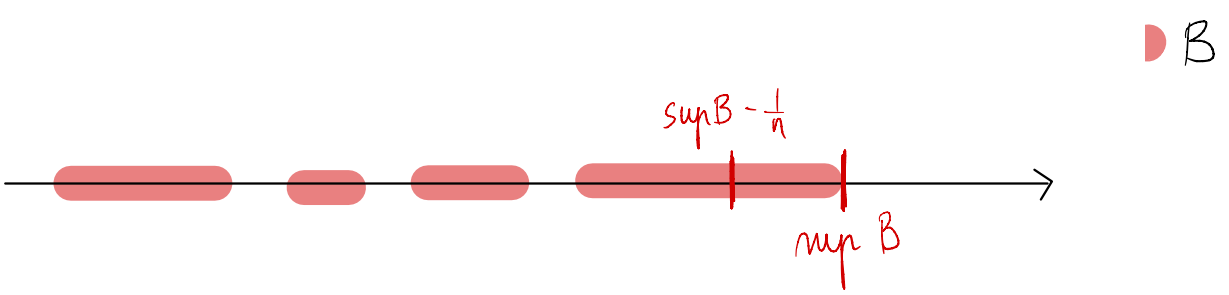


Maximizing sequences

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$$x_n \xrightarrow[n \to \infty]{} \sup B.$$





Proof:

For each $n \in \mathbb{N}$, the set $B \cap [\sup B - \frac{1}{n}, \sup B]$ is nonempty (otherwise, $\sup B - \frac{1}{n}$ would be a lower upper bound).

Let $x_n \in B \cap [\sup B - \frac{1}{n}, \sup B]$.

Then $|x_n - \sup B| \leq \frac{1}{n} \longrightarrow 0$ as $n \to \infty$.

Example: There exists a sequence of rational numbers $\{x_n\}_{n\in\mathbb{N}}$ converging to $\sqrt{2}$.

Indeed, let $B = \{x \in \mathbb{Q} : x^2 < 2\}$. Then $\sup B = \sqrt{2}$.

Let $\{x_n\}_n$ be a maximizing sequence. Then $x_n \in B \subseteq \mathbb{Q}$ for all $n \in \mathbb{N}$, and $x_n \xrightarrow[n\to\infty]{} \sup B = \sqrt{2}$.

QUESTIONS?

COMMENTS?