COMPLETENESS

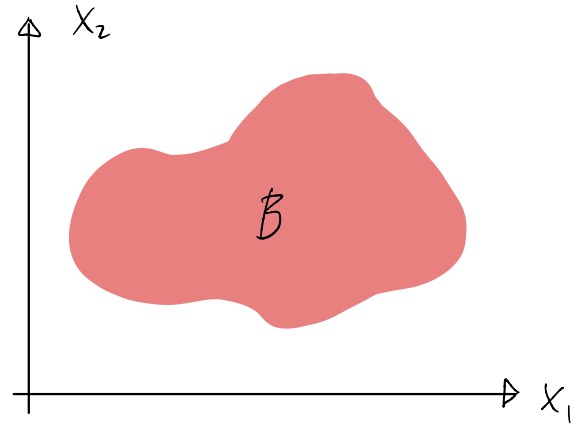## (IN R AND R^m)

The completeness principle says that $\mathbb{R}$ "fills the holes" of $\mathbb{Q}$:

If $B = \{x \in \mathbb{Q} : x^2 < 2\}$ then $\sup B = \sqrt{2} \notin \mathbb{Q}$

● B

$\sqrt{2}$

X

How can this be generalized to $\mathbb{R}^n$?

X_2

B

X_1

Cauchy sequences

We can "approximate" $\sqrt{2}$ in $\mathbb{Q}$:

$$1$$
$$1.4$$
$$1.41$$
$$1.414$$
$$1.4142$$
$$1.41421$$
$$1.414213$$
$$\vdots$$

$$\sqrt{2}$$

Cauchy sequences

We can "approximate" $\sqrt{2}$ in $\mathbb{Q}$:

$x_1 = 1$

$x_2 = 1.4$

$x_3 = 1.41$

$x_4 = 1.414$

$x_5 = 1.4142$

$x_6 = 1.41421$

$x_7 = 1.414213$

$\vdots$

$\sqrt{2}$

Cauchy sequences

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$\vdots$

$\sqrt{2}$

A sequence $\{x_n\}_{n \in \mathbb{N}}$ in $\mathbb{R}$ is <u>Cauchy</u> if for every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon \quad \text{whenever } n, m \geq N$$

<u>Theorem</u>: If $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ then there is some $x \in \mathbb{R}$ such that $x_n \xrightarrow[n \to \infty]{} x$.

Cauchy sequences in $\mathbb{R}^m$

A sequence $\{x_n\}_{n \in \mathbb{N}}$ in $\mathbb{R}^m$ is __Cauchy__ if for every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \varepsilon \qquad \text{whenever} \quad n, m \geq N$$

__Theorem__: For every Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$ in $\mathbb{R}^m$ there is some $x \in \mathbb{R}^m$ such that $x_n \xrightarrow[n \to \infty]{} x$.

__Idea of proof__: If $x_n = \begin{pmatrix} x_n^1 \\ \vdots \\ x_n^m \end{pmatrix}$ then $\{x_n\}_n$ is Cauchy in $\mathbb{R}^m$

iff each component sequence $\{x_n^1\}_n, \{x_n^2\}_n, \dots, \{x_n^m\}_n$ is Cauchy in $\mathbb{R}$.

More generally, a space is complete if every Cauchy sequence is convergent.

QUESTIONS?

COMMENTS?