FOURIER SERIES

INTRODUCTION AND MOTIVATION

Main questions:

Let $f: \mathbb{R} \to \mathbb{R}$ be given.

- Which f can we represent in the form

$$f(x) = \sum_{n=0}^{\infty} a_n \sin(nx) \quad \text{or} \quad f(x) = \sum_{n=0}^{\infty} b_n \cos(nx)$$

(or a combination of both)?

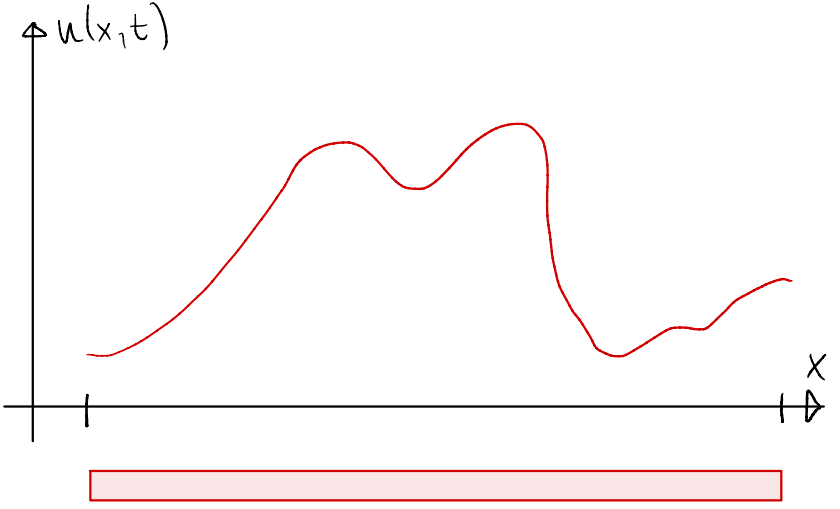
- What are a_n, b_n?
- In what sense do the series converge?

Motivation

Joseph Fourier (1768 – 1830) studied the conduction of heat in a material.

He derived the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$



He reduced the problem to finding all solutions of

$$(*) \qquad v''(x) = -\lambda v(x) \qquad \text{for any } \lambda \in \mathbb{R}.$$

If $\lambda \geq 0$ then solutions of $(*)$ are

$$v(x) = \cos(\sqrt{\lambda}\, x), \qquad v(x) = \sin(\sqrt{\lambda}\, x),$$

and any linear combination of these:

$$v(x) = a\cos(\sqrt{\lambda}\, x) + b\sin(\sqrt{\lambda}\, x) \qquad \text{for } a, b \in \mathbb{R}$$

Boundary conditions yield $\lambda = n^2$ for $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$:

$$v_n(x) = a_n \cos(nx) + b_n \sin(nx).$$

Any linear combination of these is a solution:

$$v(x) = \sum_{n=0}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right) \qquad \text{for } a_n, b_n \in \mathbb{R},\ n \in \mathbb{N}_0$$

(we don't care at the moment whether the series converges)

Recall: $\cos(y) = \dfrac{e^{iy} + e^{-iy}}{2}$, $\sin(y) = \dfrac{e^{iy} - e^{-iy}}{2i}$

$$\Rightarrow \quad v(x) = a_0 + \sum_{n=1}^{\infty} a_n \left(\frac{e^{inx} + e^{-inx}}{2} \right) + \sum_{n=1}^{\infty} b_n \left(\frac{e^{inx} - e^{-inx}}{2i} \right)$$

$$= \sum_{n=-\infty}^{\infty} \alpha_n e^{inx}$$

Exercise: Find α_n $(n \in \mathbb{Z})$ in terms of a_n, b_n $(n \in \mathbb{N}_0)$.

In order to satisfy <u>initial conditions</u>, Fourier needed to find coefficients $\alpha_n \in \mathbb{C}$ such that $v = f$ for an arbitrary function f:

$$\sum_{n=-\infty}^{\infty} \alpha_n e^{inx} \overset{!}{=} f(x) \qquad \forall\ x$$

<u>Note:</u>
Each function $e_n(x) = e^{inx}$ $(n \in \mathbb{Z})$ is 2π-periodic:

$$e_n(x+2\pi) = e_n(x) \qquad \forall\ x \in \mathbb{R}.$$

Therefore, we restrict x to the interval $x \in [-\pi, \pi]$.

Exercise

Prove that $\int_{-\pi}^{\pi} e_n(x)\overline{e_m(x)}\,dx = \begin{cases} 2\pi & (n=m) \\ 0 & (n\neq m) \end{cases}$

(where $e_n(x) = e^{inx}$).

Therefore, $\{e_n\}_{n\in\mathbb{Z}}$ is <u>orthonormal</u>:

$$\langle e_n, e_m \rangle_{L^2} = \delta_{n,m} \qquad \forall\ n,m \in \mathbb{Z}$$

where $\delta_{n,m} = \begin{cases} 1 & (n=m) \\ 0 & (n\neq m) \end{cases}$ (the Krönecker delta) and

$$\langle f, g \rangle_{L^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$$ (the L^2 inner product).

If we want $\sum_{n\in\mathbb{Z}} \alpha_n e_n = f$ then we get

$$\langle f, e_m \rangle_{L^2} = \left\langle \sum_{n\in\mathbb{Z}} \alpha_n e_n, e_m \right\rangle_{L^2} = \sum_{n\in\mathbb{Z}} \alpha_n \langle e_n, e_m \rangle_{L^2}$$

$$= \sum_{n\in\mathbb{Z}} \alpha_n \delta_{n,m} = \alpha_m .$$

Hence, $f(x) = \sum_{n\in\mathbb{Z}} \alpha_n e^{inx}$, $\alpha_n = \langle f, e_n \rangle_{L^2}$

Definition: The above series is the Fourier series of f

Warning: It is not obvious that this series converges!

By $\sum_{n=-\infty}^{\infty} \alpha_n e_n$ we mean $\lim_{N\to\infty} f_N$, where

$$(**) \qquad f_N(x) = \sum_{n=-N}^{N} \alpha_n e_n(x) \qquad \text{for } x \in \mathbb{R}$$

Definition

A function of the form $(**)$, for $\alpha_{-N}, \dots, \alpha_N \in \mathbb{C}$, is a **trigonometric polynomial**

- "trigonometric" since $e_n(x) = e^{inx} = \cos(nx) + i\sin(nx)$
- "polynomial" since $e_n(x) = e^{inx} = (e^{ix})^n$

Example:
Compute the Fourier series of $f(x) = x \quad (x \in [-\pi, \pi])$.

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$n = 0$: $\alpha_0 = \langle f, e_0 \rangle_{L^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \, \overline{e^{0 \cdot nx}} \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \, dx = 0$

$n \neq 0$: $\alpha_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \, \overline{e^{inx}} \, dx = \frac{-1}{2\pi} \left[\frac{x e^{-inx}}{in} \right]_{x=-\pi}^{\pi} + \frac{1}{2in\pi} \underbrace{\int_{-\pi}^{\pi} e^{-inx} \, dx}_{=0}$

$$= \frac{-1}{2\pi in} \left(\pi e^{-in\pi} + \pi e^{-in\pi} \right) = - \frac{\cos(n\pi)}{in}$$

$$= \frac{(-1)^{n+1}}{in}.$$

Hence, the Fourier series of f is $\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{i(-1)^n}{n} e^{inx}$.

We can convert this complex series to a real series:

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{(-1)^{n+1}}{in} e^{inx} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{in} e^{inx} + \sum_{n=1}^{\infty} \frac{(-1)^{-n+1}}{-in} e^{-inx}$$

$$= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \left(\frac{e^{inx} - e^{-inx}}{2i} \right) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$$

If f is real-valued, then we can <u>always</u> convert a (complex) Fourier series into a real sine/cosine series in this way.

It's not obvious that the series $\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{i(-1)^n}{n} e^{inx}$ or

$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$ converge!

For instance,

$$\sum_{n=1}^{\infty} \left| \frac{2(-1)^{n+1}}{n} \sin(nx) \right| \leq \sum_{n=1}^{\infty} \frac{2}{n} = \infty$$

so standard tricks will not yield convergence.

QUESTIONS ?

COMMENTS ?