FOURIER CONVERGENCE I

THE MATHEMATICAL FRAMEWORK

Recall

- We let $f : [-\pi, \pi] \to \mathbb{R}$ be given
- The Fourier series of f is

$$\sum_{n \in \mathbb{Z}} \alpha_n e_n, \qquad \alpha_n = \langle f, e_n \rangle_{L^2}$$

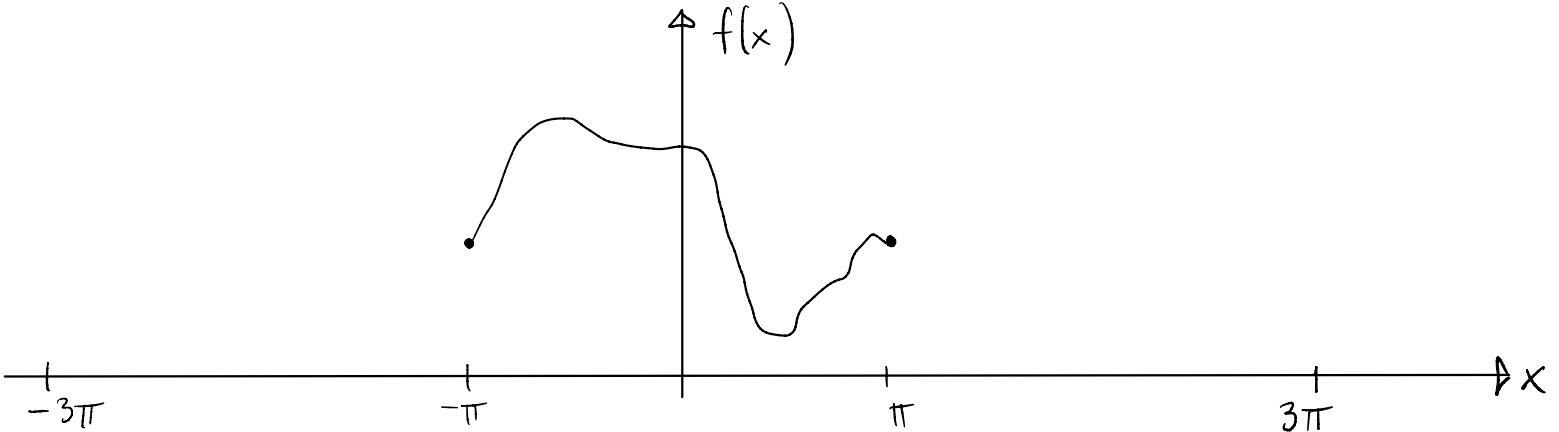
where $e_n(x) = e^{inx}$ and $\langle f, g \rangle_{L^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx$

- We hope that $f = \sum_{n \in \mathbb{Z}} \alpha_n e_n$, but need to prove that this series converges!

Want: $f = \sum_{n \in \mathbb{Z}} \alpha_n e_n$, where $\alpha_n = \langle f, e_n \rangle_{L^2}$ (1)

Each function e_n is 2π-periodic, so f must necessarily also be 2π-periodic.

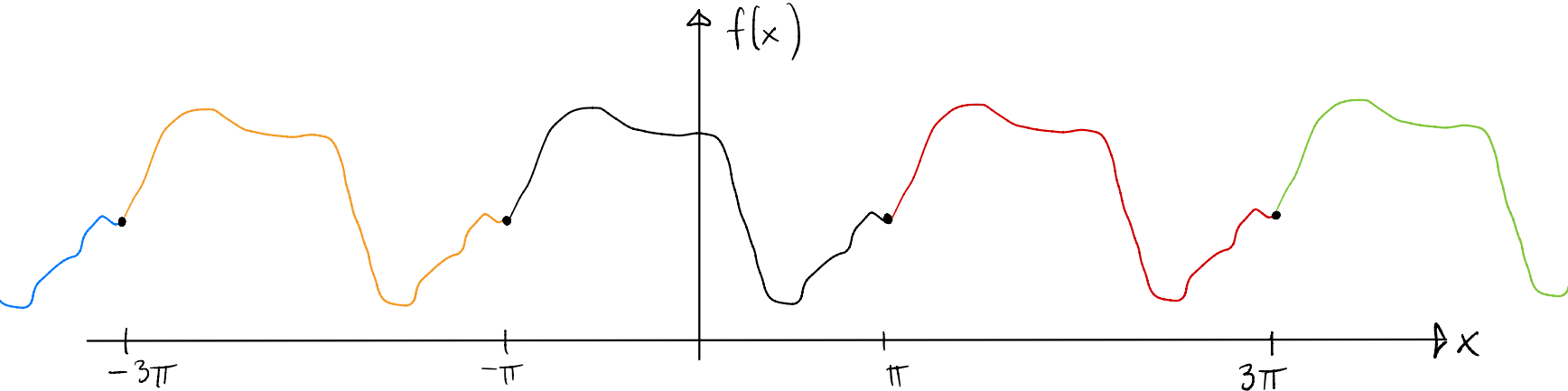
Definition: $C_p = \{ f \in C([-\pi, \pi], \mathbb{C}) : f(-\pi) = f(\pi) \}$



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Exercise: $\langle \cdot, \cdot \rangle_{L^2}$ is an inner product on $C([-\pi, \pi], \mathbb{C})$.

Definition

The L^2 norm of $f : [-\pi, \pi] \to \mathbb{C}$ is

$$\|f\|_{L^2} = \sqrt{\langle f, f \rangle_{L^2}} = \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx}$$

The L^2 norm is also meaningful for discontinuous f.

We will allow finitely many discontinuities.

Definition

Let D denote the set of all 2π-periodic $f : \mathbb{R} \to \mathbb{C}$ for which there are points $a_1, \dots, a_n \in [-\pi, \pi)$ with $a_1 < a_2 < \dots < a_n$ such that:

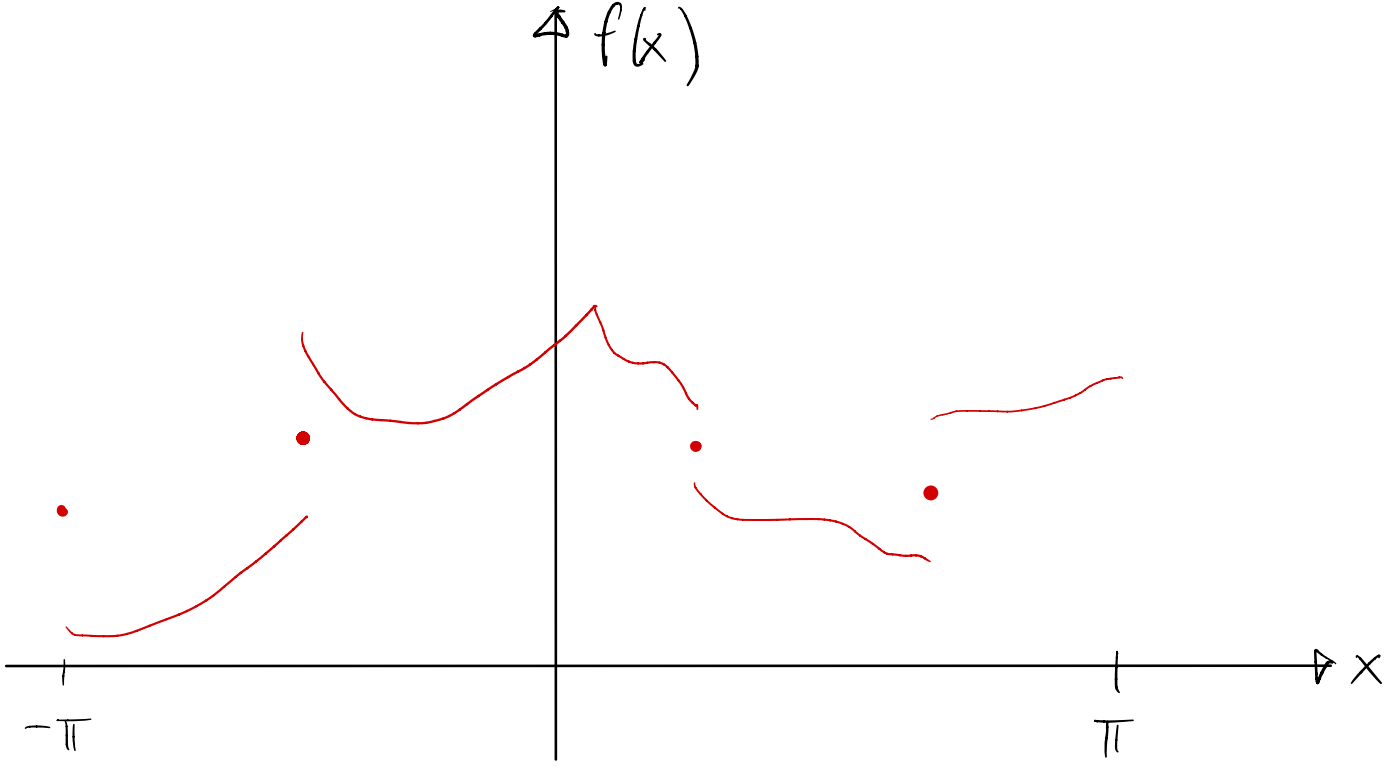
(i) f is continuous in $(a_1, a_2), \dots, (a_{n-1}, a_n), (a_n, a_1 + 2\pi)$

(ii) the limits $f^-(a_i) = \lim_{h \searrow 0} f(a_i - h)$ and $f^+(a_i) = \lim_{h \searrow 0} f(a_i + h)$ exist for all $i = 1, \dots, n$

(iii) $f(a_i) = \dfrac{f^-(a_i) + f^+(a_i)}{2}$ for all $i = 1, \dots, n$.

Note: C_p is a closed subspace of D (with respect to $\|\cdot\|_{L^2}$)

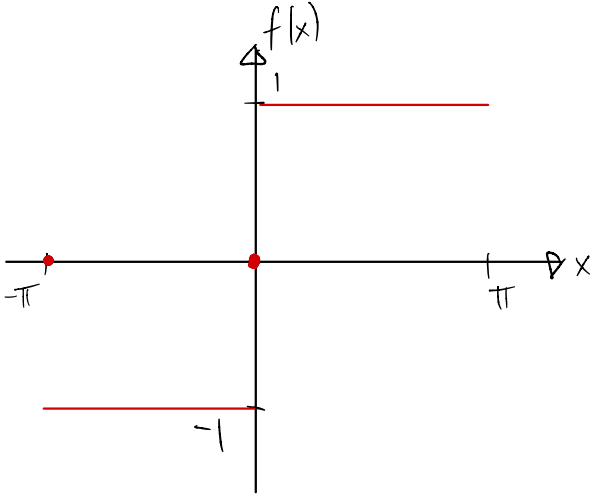
A "typical" $f \in D$:

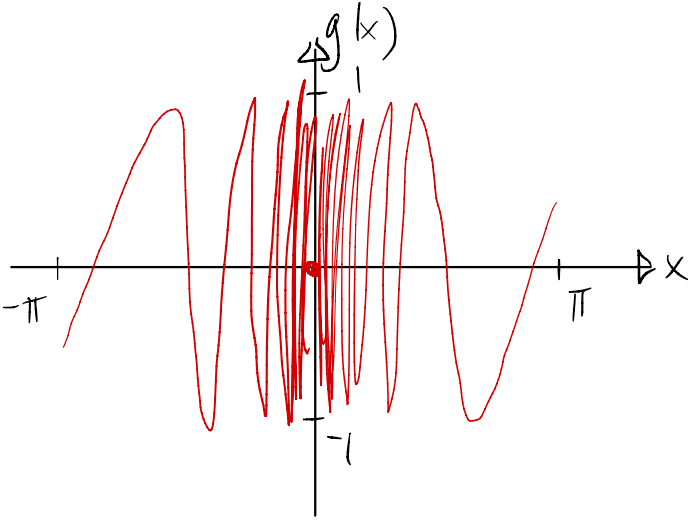


Example

Let $f(x) = \begin{cases} -1 & (-\pi < x < 0) \\ 0 & (x = 0,\ x = -\pi) \\ 1 & (0 < x < \pi) \end{cases}$ and $g(x) = \begin{cases} \sin(1/x) & (x \neq 0) \\ 0 & (x = 0). \end{cases}$

Then $f \in D$, but $g \notin D$.





Note: We can make sense of the integral of f and of fg for any $f, g \in D$

Exercise: $\langle \cdot, \cdot \rangle_{L^2}$ is an inner product on D.

We have now two norms on D:

$$\|f\|_{L^\infty} = \sup\{|f(x)| : x \in [-\pi, \pi]\}$$

$$\|f\|_{L^2} = \sqrt{\langle f, f\rangle_{L^2}} = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2}$$

Exercise: Let $f, f_1, f_2, \ldots \in D$.

(i) If $f_n \xrightarrow[n\to\infty]{} f$ in $\|\cdot\|_{L^\infty}$ then $f_n \xrightarrow[n\to\infty]{} f$ in $\|\cdot\|_{L^2}$

(ii) The opposite is not true.

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(ii) The opposite is not true.

Hint for (ii): Take $f_n(x) = \begin{cases} 1 & |x| < \frac{1}{n} \\ 0 & \text{else} \end{cases}$ $(x \in [-\pi,\pi])$ and $f \equiv 0$.

QUESTIONS?

COMMENTS?

Next video: Main goals and some first steps.