# FOURIER CONVERGENCE II

## MAIN GOALS AND FIRST STEPS

Recall:

- $C_p = \{$ continuous, $2\pi$-periodic functions $\}$
- $D = \{$ functions with finitely many discontinuities $\}$
- The Fourier series of $f \in D$ is

$$\sum_{n \in \mathbb{Z}} \alpha_n e_n, \qquad \alpha_n = \langle f, e_n \rangle$$

where $e_n(x) = e^{inx}$ and $\langle f, g \rangle_{L^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx$

Recall:

- $\|f\|_{L^\infty} = \sup\{|f(x)| : x \in [-\pi, \pi]\}$
- $\|f\|_{L^2} = \sqrt{\langle f, f \rangle_{L^2}} = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2}$
- A <u>trigonometric polynomial</u> is a function of the form $\sum_{n=-N}^{N} \alpha_n e_n$ for some $N \in \mathbb{N}$ and $\alpha_{-N}, \dots, \alpha_N \in \mathbb{C}$.

## Main question

For which functions $f$ does the Fourier series

$$\sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n$$

converge, and in what sense?

We will not be able to prove that the Fourier series converges uniformly (in $\|\cdot\|_{L^\infty}$) to $f$ for every continuous $f$. (In fact, it is not true!)

**Definition**

Let $f, f_1, f_2, \ldots : X \to \mathbb{R}$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ converges (pointwise/uniformly) in Cesàro mean to $f$ if

$$S_N \xrightarrow[N\to\infty]{} f \quad \text{pointwise/uniformly}$$

where $S_N(x) = \frac{1}{N}\sum_{n=1}^{N} f_n(x)$.

Main result 1

Let $f \in D$ and let $f_N = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n$. Then $\{f_N\}_{N \in \mathbb{N}}$ converges to $f$ pointwise in Cesàro mean, and uniformly in Cesàro mean in intervals $[a, b]$ where $f$ is continuous.

We will prove this in a later video!

## Main Result 2

The set {all trig. pol.} is dense in $C_p$ with respect to $\|\cdot\|_{L^\infty}$.

In other words, for every $f \in C_p$ there are trigonometric polynomials $P_1, P_2, \ldots$ so that

$$\|P_N - f\|_{L^\infty} \xrightarrow[N\to\infty]{} 0.$$

Note: The trig. pol. might not be the partial Fourier series of $f$!

## Main result 3

Let $f \in D$ and let $f_N = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n$.

Then $\|f - f_N\|_{L^2} \xrightarrow[N \to \infty]{} 0$.

Assume now

**Main result 1**

Let $f \in D$ and let $f_N = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n$. Then $\{f_N\}_{N \in \mathbb{N}}$ converges to $f$ pointwise in Cesàro mean, and uniformly in Cesàro mean in intervals $[a, b]$ where $f$ is continuous.

We will now prove

Main result 1 $\Rightarrow$ Main result 2 $\Rightarrow$ Main result 3

## Main Result 2

For every $f \in C_p$ there are trigonometric polynomials $p_1, p_2, \ldots$ so that $\|p_N - f\|_{L^\infty} \xrightarrow[N\to\infty]{} 0$.

Proof:

Let $f \in C_p$, $f_n = \sum_{k=-n}^{n} \langle f, e_k \rangle e_k$, $p_N = \frac{1}{N} \sum_{n=1}^{N} f_n$.

Each function $p_N$ is a trigonometric polynomial, and by Main Result 1, $p_N \xrightarrow[N\to\infty]{} f$ uniformly. ∎

To prove Main Result 3, we will need some lemmas.

**Lemma 1**

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space, let $u \in V$ and let $\{e_1, \dots, e_N\}$ be orthonormal (i.e. $\langle e_n, e_m \rangle = \delta_{n,m}$).

Then $p = \sum_{n=1}^{N} \langle u, e_n \rangle e_n$ satisfies

$$\|p - u\| = \inf \{ \|q - u\| : q \in \operatorname{span}\{e_1, \dots, e_N\} \}$$

**Proof:**

$$\|p-u\|^2 = \langle p, p \rangle - \langle p, u \rangle - \langle u, p \rangle + \langle u, u \rangle$$

$$= \sum_{n,m=1}^{N} \langle u, e_n \rangle \overline{\langle u, e_m \rangle} \langle e_n, e_m \rangle - 2 \sum_{n=1}^{N} \langle u, e_n \rangle \overline{\langle u, e_n \rangle} + \|u\|^2$$

$$= \sum_{n=1}^{N} |\langle u, e_n \rangle|^2 - 2 \sum_{n=1}^{N} |\langle u, e_n \rangle|^2 + \|u\|^2$$

$$= \|u\|^2 - \sum_{n=1}^{N} |\langle u, e_n \rangle|^2$$

**Lemma 1**

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space, let $u \in V$ and let $\{e_1, \dots, e_N\}$ be orthonormal (i.e. $\langle e_n, e_m \rangle = \delta_{n,m}$).

Then $p = \sum_{n=1}^{N} \langle u, e_n \rangle e_n$ satisfies

$$\|p - u\| = \inf \{ \|q - u\| : q \in \operatorname{span}\{e_1, \dots, e_N\} \}$$

**Proof:** Next, let $q \in \operatorname{span}\{e_1, \dots, e_N\}$, so $q = \alpha_1 e_1 + \cdots + \alpha_N e_N$. Then

$$\begin{aligned}
\|q - u\|^2 &= \|u\|^2 - \sum_{n=1}^{N} \left( \alpha_n \langle e_n, u \rangle + \overline{\alpha_n} \langle u, e_n \rangle \right) + \sum_{n=1}^{N} |\alpha_n|^2 \\
&= \|u\|^2 + \sum_{n=1}^{N} |\alpha_n - \langle u, e_n \rangle|^2 - \sum_{n=1}^{N} |\langle u, e_n \rangle|^2 \\
&= \|p - u\|^2 + \sum_{n=1}^{N} |\alpha_n - \langle u, e_n \rangle|^2 \geq \|p - u\|^2.
\end{aligned}$$

∎

**Lemma 2**

Let $f \in C_p$ and $f_N = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n$. Then $\|f_N - f\|_{L^2} \xrightarrow[N \to \infty]{} 0$.

Proof: Let $\varepsilon > 0$. We want to show there is some $M \in \mathbb{N}$ so that $\|f - f_N\|_{L^2} < \varepsilon$ when $N \geq M$.

By Main Result 2, there is a trig. pol. $p = \sum_{n=-M}^{M} \alpha_n e_n$ so that $\|f - p\|_{L^2} < \varepsilon$.

Now let $N \geq M$. Then $p \in \text{span}\{e_{-N}, \ldots, e_N\}$, so by Lemma 2,

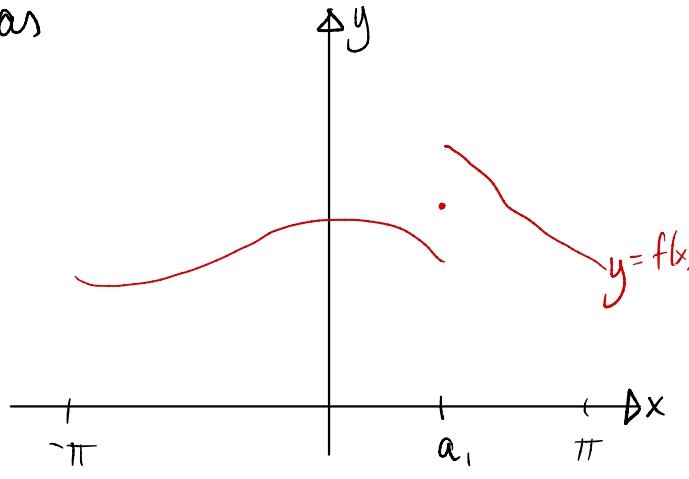
$$\|f - f_N\|_{L^2} = \left\| f - \sum_{n=-N}^{N} \langle f, e_n \rangle e_n \right\|_{L^2} \leq \|f - p\|_{L^2} < \varepsilon.$$



**Lemma 3**

Let $f \in D$ and $\varepsilon > 0$. Then there is some $g \in C_p$ such that $\|f - g\|_{L^2} < \varepsilon$.

**Proof**: Assume for simplicity that $f$ has one discontinuity, at $a_1 \in (-\pi, \pi)$.



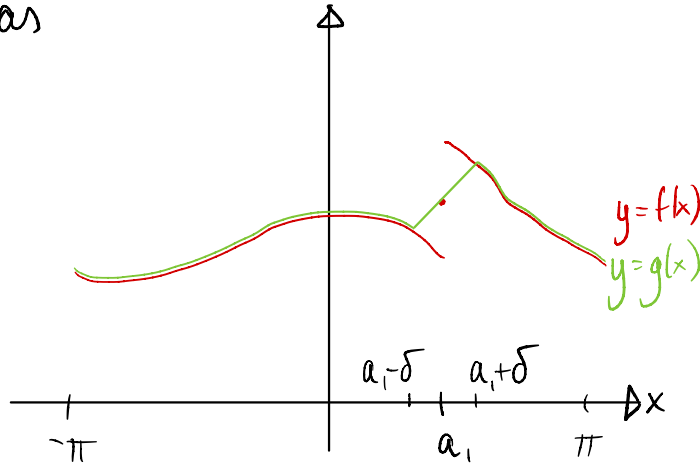
**Lemma 3**

Let $f \in D$ and $\varepsilon > 0$. Then there is some $g \in C_P$ such that $\|f - g\|_{L^2} < \varepsilon$.

**Proof**: Assume for simplicity that $f$ has one discontinuity, at $a_1 \in (-\pi, \pi)$.

Let $\delta = \dfrac{\varepsilon^2}{9\|f\|_{L^\infty}^2}$ and define $g \in C_P$ by

$$g(x) = \begin{cases} f(x) & \text{(if } |x - a_1| \geq \delta\text{)} \\ f(a_1 - \delta) + \dfrac{x - a_1 + \delta}{2\delta}\left(f(a_1 + \delta) - f(a_1 - \delta)\right) & \text{(if } |x - a_1| < \delta\text{)} \end{cases}$$



**Lemma 3**

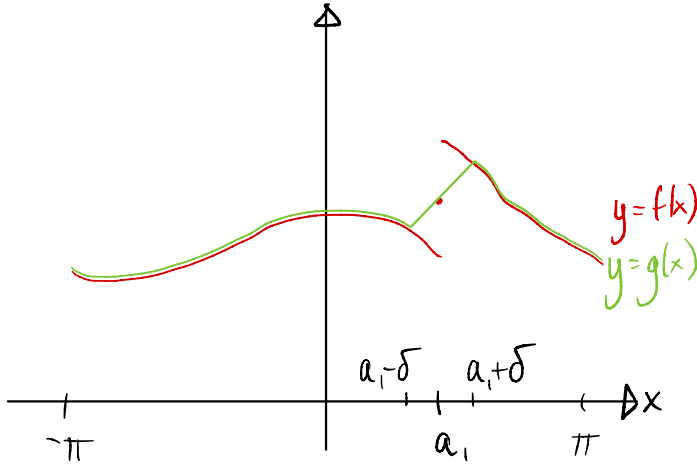
Let $f \in D$ and $\varepsilon > 0$. Then there is some $g \in C_p$ such that $\|f-g\|_{L^2} < \varepsilon$.

**Proof**: Then

$$\|f-g\|_{L^2}^2 = \int_{a_1-\delta}^{a_1+\delta} |f(x)-g(x)|^2 dx$$

$$\leq 2\delta \cdot \left(2\|f\|_{L^\infty}\right)^2 = 8\|f\|_{L^\infty}^2 \, \frac{\varepsilon^2}{9\|f\|_{L^\infty}^2}$$

$$< \varepsilon^2.$$




∎

## Main result 3

Let $f \in D$ and let $f_N = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n$.

Then $\|f - f_N\|_{L^2} \xrightarrow[N \to \infty]{} 0$.

**Proof:** Let $\varepsilon > 0$. Let $g \in C_p$ be s.t. $\|f - g\|_{L^2} < \varepsilon$.

Let $M \in \mathbb{N}$ be s.t. $\|g - g_N\|_{L^2} < \varepsilon \quad \forall N \geq M$.

Finally, note that $\|f_N - g_N\|_{L^2} \leq \|f - g\|_{L^2}$ (exercise!).

Then, for every $N \geq M$,

$$\|f - f_N\|_{L^2} \leq \|f - g\|_{L^2} + \|g - g_N\|_{L^2} + \|g_N - f_N\|_{L^2} < 3 \cdot \varepsilon.$$



QUESTIONS?
COMMENTS?

Next videos: Prove Main Result 1 - convergence in Cesàro mean.