THE DIRICHLET KERNEL

Recall:

- The Fourier series of f is

$$\sum_{n=-\infty}^{\infty} \langle f, e_n \rangle_{L^2} e_n$$

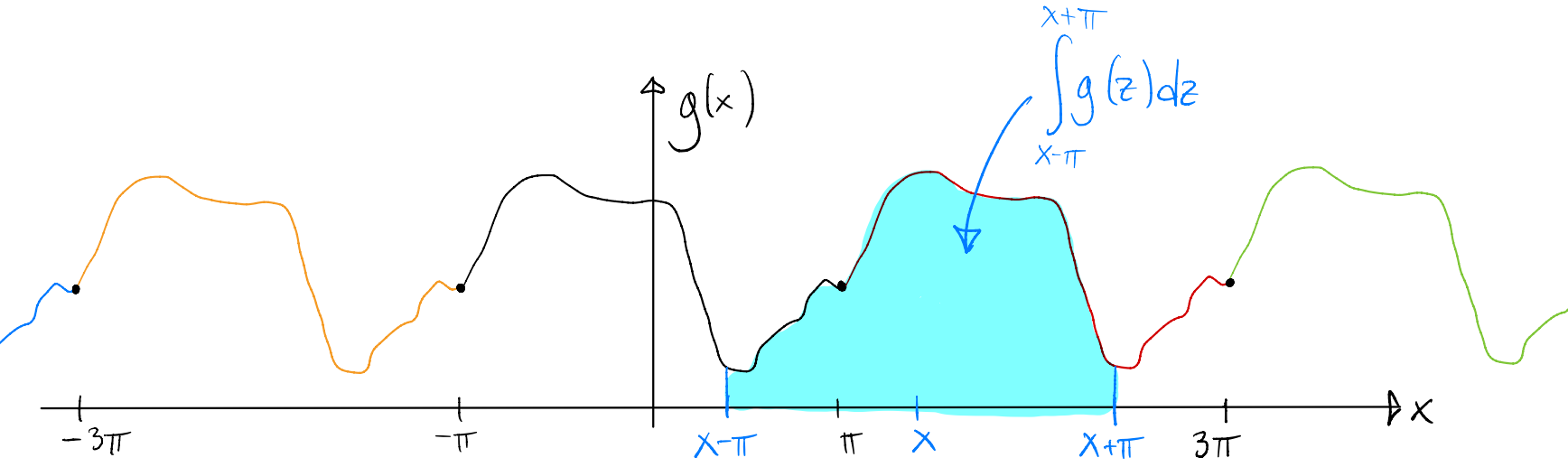
where $e_n(x) = e^{inx}$ and $\langle f, g \rangle_{L^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx$.

- By convergence of the Fourier series we mean the convergence (pointwise/uniformly/in L^2/in Cesàro mean) of the partial sum

$$f_N(x) = \sum_{n=-N}^{N} \langle f, e_n \rangle_{L^2} e_n$$

Note: If $g : \mathbb{R} \to \mathbb{R}$ is 2π-periodic ($g(z+2\pi) = g(z)\ \forall z$) then

$$\int_{x-\pi}^{x+\pi} g(z)\,dz = \int_{-\pi}^{\pi} g(z)\,dz \qquad \text{for any } x \in \mathbb{R}.$$



If f is 2π-periodic and $x \in [-\pi, \pi)$ then

$$\langle f, e_n \rangle e_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny}\, dy\, e^{inx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{in(x-y)}\, dy$$

$$(u = x-y) \quad = \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} \underbrace{f(x-u)\, e^{inu}}_{2\pi\text{-periodic}}\, du = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u) e^{inu}\, du.$$

Hence,

$$f_N(x) = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u) \underbrace{\left(\sum_{n=-N}^{N} e^{inu} \right)}_{= D_N(u)} du$$

$$\Longrightarrow \quad f_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u)\, D_N(u)\, du$$

The functions $D_N(x) = \sum_{n=-N}^{N} e^{inx}$ are the Dirichlet kernels

Remark

An integral of the form $\int f(x-u)\, g(u)\, du$ is the convolution of f and g, written $f * g$.

Hence, $f_N = \frac{1}{2\pi} f * D_N$.

$$\Longrightarrow \quad f_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u)\, D_N(u)\, du$$

The functions $D_N(x) = \sum_{n=-N}^{N} e^{inx}$ are the

Dirichlet kernels

Remark:

We can write $D_N(x) = \sum_{n=-N}^{N} (e^{ix})^n$, a "geometric series", but $|e^{ix}| = 1$, so the series might not converge as $N \to +\infty$.

Use formula for geometric sums:

$$D_N(x) = \sum_{n=-N}^{-1} e^{ixn} + \sum_{n=0}^{N} e^{ixn} = \sum_{n=1}^{N} \left(e^{-ix}\right)^n + \sum_{n=0}^{N} \left(e^{ix}\right)^n$$

$$= \frac{e^{-ix} - \left(e^{-ix}\right)^{N+1}}{1 - e^{-ix}} + \frac{1 - \left(e^{ix}\right)^{N+1}}{1 - e^{ix}}$$

$$= \frac{1}{e^{ix/2} - e^{-ix/2}} \left(e^{-ix/2} - e^{-(N+1/2)ix} - e^{-ix/2} + e^{(N+1/2)ix} \right)$$

$$= \frac{\sin\left((N+1/2)x\right)}{\sin\left(x/2\right)}$$

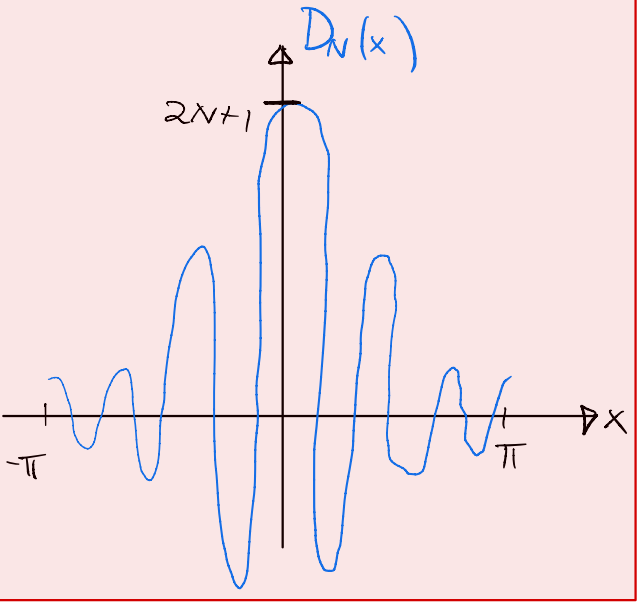
Hence:

$$f_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u) \underbrace{\frac{\sin\left((N+1/2)u\right)}{\sin(u/2)}}_{= D_N(u)} du$$

But the Dirichlet kernel is very badly behaved!

- $|D_N(x)| \leq |D_N(0)| = 2N+1 \xrightarrow[N\to\infty]{} \infty$
- $\int_{-\pi}^{\pi} |D_N(x)|\,dx \xrightarrow[N\to\infty]{} \infty$

  even though $\int_{-\pi}^{\pi} D_N(x)\,dx = 2\pi$
- It oscillates heavily



All of this makes it very challenging to prove convergence of

$$f_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u)\, D_N(u)\, du$$

as $N \to \infty$.

QUESTIONS?

COMMENTS?

Next video: Will instead prove Cesàro convergence of $\{f_N\}_N$.