CESÀRO CONVERGENCE OF FOURIER SERIES

Recall:

- $f_N(x) = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u) D_N(u)\, du$

  where $D_N(u) = \sum_{n=-N}^{N} e^{inu} = \frac{\sin\left((N+1/2)u\right)}{\sin(u/2)}$

- D_N is badly behaved

We will instead prove Cesàro convergence:

Definition

Let $f, f_1, f_2, \ldots : X \to \mathbb{R}$. The sequence $\{f_n\}_{n \in \mathbb{N}}$ converges (pointwise/uniformly) in Cesàro mean to f if

$$S_N \xrightarrow[N \to \infty]{} f \quad \text{pointwise/uniformly}$$

where $S_N(x) = \frac{1}{N} \sum_{n=1}^{N} f_n(x)$.

We have

$$S_N(x) = \frac{1}{N}\sum_{n=0}^{N-1} f_n(x) = \frac{1}{N}\sum_{n=0}^{N-1} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\,D_n(u)\,du$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\left(\frac{1}{N}\sum_{n=0}^{N-1} D_n(u)\right) du$$

$$= F_N(u)$$

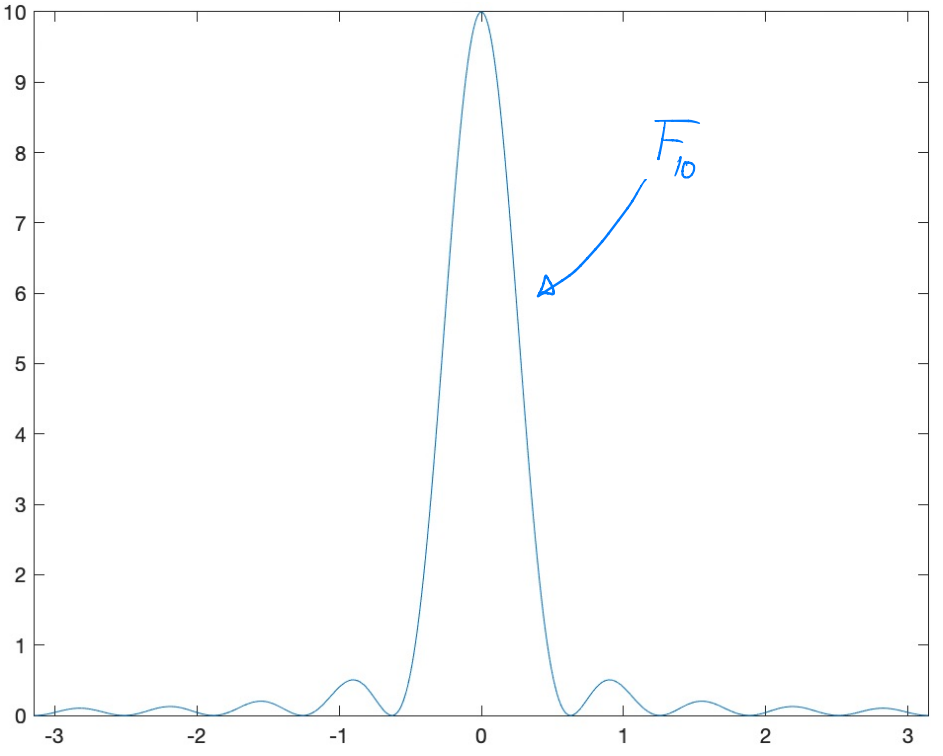
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\,F_N(u)\,du$$

F_N is the $(N\text{-th})$ Fejér kernel

Lemma

$$F_N(x) = \begin{cases} N & (x = 0) \\ \dfrac{\sin\left(\frac{Nx}{2}\right)^2}{N \sin\left(\frac{x}{2}\right)^2} & (x \neq 0) \end{cases}$$

Proof: Apply some trigonometric identities. ∎



Lemma

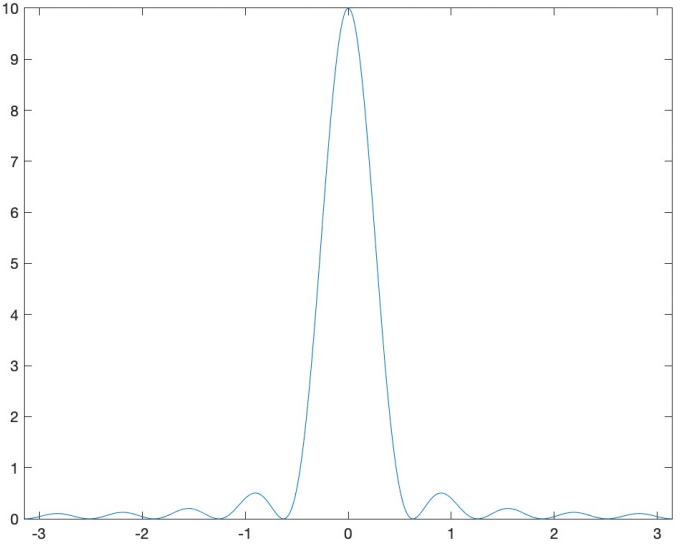
F_N is even, nonnegative, and satisfies $\int_{-\pi}^{\pi} F_N(x)\,dx = 2\pi$ and $F_N(x) \le \frac{\pi^2}{Nx^2}$ for all nonzero $x \in [-\pi, \pi]$

Proof

$F_N(x) = \frac{\sin\left(Nx/2\right)^2}{N\sin\left(x/2\right)^2}$ is clearly even and nonnegative. Next,

$$\int_{-\pi}^{\pi} F_N(x)\,dx = \frac{1}{N}\sum_{n=0}^{N-1} \int_{-\pi}^{\pi} D_n(x)\,dx$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} 2\pi = 2\pi$$



Lemma

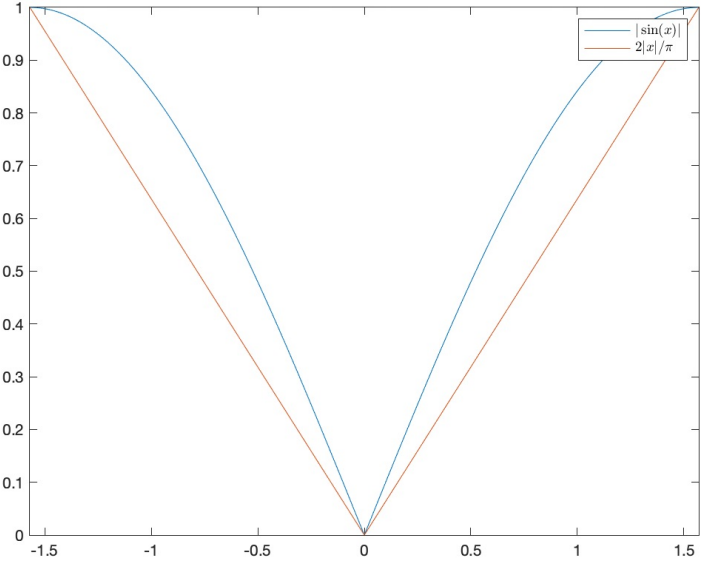
F_N is even, nonnegative, and satisfies $\int_{-\pi}^{\pi} F_N(x)\,dx = 2\pi$ and $F_N(x) \le \frac{\pi^2}{Nx^2}$ for all nonzero $x \in [-\pi, \pi]$

For the last estimate, note that

- $|\sin(x)| \le 1 \quad \forall\, x \in \mathbb{R}$
- $|\sin(x)| \ge \frac{2|x|}{\pi} \quad \forall\, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

so

$$F_N(x) = \frac{\sin\left(\frac{Nx}{2}\right)^2}{N\sin\left(\frac{x}{2}\right)^2} \le \frac{1}{N\cdot\left(\frac{2\left|\frac{x}{2}\right|}{\pi}\right)^2} = \frac{\pi^2}{Nx^2}.$$



Lemma

Let $f \in D$. For every $x \in [-\pi, \pi]$ and $\varepsilon > 0$ there is some $\delta > 0$ such that $\left| f(x) - \frac{f(x-u) + f(x+u)}{2} \right| < \varepsilon \quad \forall\, |u| < \delta$.

If f is continuous in $[a, b]$ then δ is independent of $x \in [a, b]$.

Proof:

If f is continuous at x then δ can be chosen as in the definition of continuity.

If f is continuous in $[a, b]$ then it is also uniformly continuous there, so δ can be chosen independently of $x \in [a, b]$.

Lemma

Let $f \in D$. For every $x \in [-\pi, \pi]$ and $\varepsilon > 0$ there is some $\delta > 0$ such that $\left| f(x) - \frac{f(x-u) + f(x+u)}{2} \right| < \varepsilon \quad \forall\, |u| < \delta$.

If f is continuous in $[a, b]$ then δ is independent of $x \in [a, b]$.

Proof:

If f is discontinuous at x then, by assumption,

$$f(x) = \frac{f^-(x) + f^+(x)}{2} = \lim_{u \searrow 0} \frac{f(x-u) + f(x+u)}{2},$$

so a $\delta > 0$ exists for which $\left| f(x) - \frac{f(x-u) + f(x+u)}{2} \right| < \varepsilon$ for all u with $|u| < \delta$.

∎

Main Result 1 (Fejér's theorem)

If $f \in D$ then $S_N \xrightarrow[N\to\infty]{} f$ pointwise in $[-\pi,\pi]$, and uniformly in every interval $[a,b]$ where f is continuous.

Proof: First, we have

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\,F_N(u)\,du = \frac{1}{4\pi}\int_{-\pi}^{\pi} f(x-u)\,F_N(u)\,du + \frac{1}{4\pi}\int_{-\pi}^{\pi} f(x-u)\,F_N(u)\,du$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi} f(x-u)\,F_N(u)\,du + \frac{1}{4\pi}\int_{-\pi}^{\pi} f(x+u)\,\underbrace{F_N(-u)}_{=F_N(u)}\,du$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(x-u)+f(x+u)}{2}\,F_N(u)\,du.$$

Main Result 1 (Fejér's Theorem)

If $f \in D$ then $S_N \xrightarrow[N\to\infty]{} f$ pointwise in $[-\pi,\pi]$, and uniformly in every interval $[a,b]$ where f is continuous.

Let $\varepsilon > 0$, $x \in [-\pi,\pi]$ and let $\delta > 0$ be as in the lemma.

$$S_N(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(x-u)+f(x+u)}{2} F_N(u)\,du - f(x)\cdot \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(u)\,du}_{=1}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{f(x-u)+f(x+u)}{2} - f(x)\right) F_N(u)\,du$$

$$= \frac{1}{2\pi}\int_{-\delta}^{\delta} \left(\frac{f(x-u)+f(x+u)}{2} - f(x)\right) F_N(u)\,du + \frac{1}{2\pi}\int_{[-\pi,\pi]\setminus[-\delta,\delta]} \left(\frac{f(x-u)+f(x+u)}{2} - f(x)\right) F_N(u)\,du .$$

Main Result 1 (Fejér's theorem)

If $f \in D$ then $S_N \xrightarrow[N\to\infty]{} f$ pointwise in $[-\pi,\pi]$, and uniformly in every interval $[a,b]$ where f is continuous.

Hence,

$$|S_N(x)-f(x)| \le \frac{1}{2\pi}\int_{-\delta}^{\delta} \mathcal{E}\, F_N(u)\,du + \frac{1}{2\pi}\int_{[-\pi,\pi]\setminus[-\delta,\delta]} \left|\frac{f(x-u)+f(x+u)}{2} - f(x)\right| F_N(u)\,du$$

$$\le \mathcal{E}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(u)\,du + \frac{2\|f\|_{L^\infty}}{2\pi}\int_{[-\pi,\pi]\setminus[-\delta,\delta]} \frac{\pi^2}{N u^2}\,dx$$

$$= \mathcal{E} + \frac{2\|f\|_{L^\infty}\pi}{N}\left(\frac{1}{\delta}-\frac{1}{\pi}\right)$$

$$\le 2\mathcal{E} \qquad \text{for } N \text{ large enough.}$$

Main Result 1 (Fejér's theorem)

If $f \in D$ then $S_N \xrightarrow[N\to\infty]{} f$ pointwise in $[-\pi, \pi]$, and uniformly in every interval $[a, b]$ where f is continuous.

Finally, if f is continuous in $[a, b]$ then the same δ works for all $x \in [a, b]$, so the estimate is independent of the choice of x.



QUESTIONS?
COMMENTS?