CONVERGENCE

Recall:

A sequence $\{x_n\}_n$ in $\mathbb{R}$ converges to $x \in \mathbb{R}$ if for every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon \quad \text{whenever} \quad n \geq N.$$

Let (X, d) be a metric space.

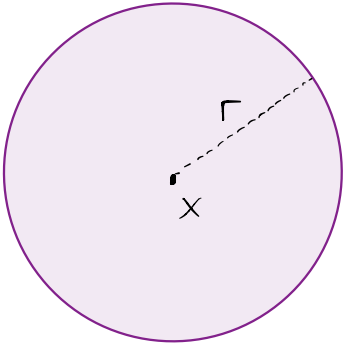
A sequence $\{x_n\}_n$ in X converges to $x \in X$ if for every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon \quad \text{whenever} \quad n \geq N.$$

$\{x_n\}_n$ converges to x iff $d(x_n, x) \longrightarrow 0$ as $n \to \infty$.

The open ball with center x and radius $r > 0$ is

$$B(x; r) = \{z \in X : d(x, z) < r\}.$$



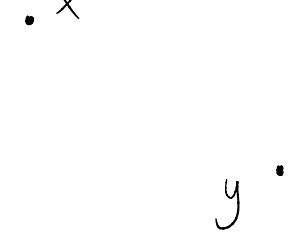
We can reformulate convergence:

$\{x_n\}_n$ converges to x if for every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that $x_n \in B(x; \varepsilon)$ whenever $n \geq N$

Proposition (the limit is unique)

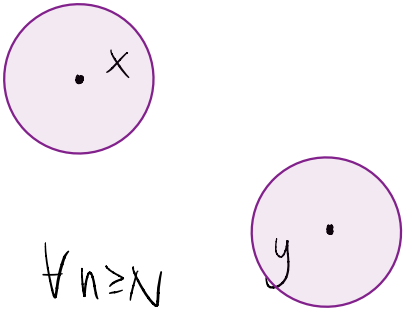
If $\{x_n\}_n$ converges to both x and y then $x = y$.

Proof: Assume $d(x,y) > 0$.



Proposition (the limit is unique)
If $\{x_n\}_n$ converges to both x and y then $x = y$.

Proof: Assume $d(x,y) > 0$.



Let $\varepsilon = \frac{d(x,y)}{4}$.

Let $N \in \mathbb{N}$ be s.t. $d(x_n, x) < \varepsilon \quad \forall n \geq N$

and $M \in \mathbb{N}$ be s.t. $d(x_n, y) < \varepsilon \quad \forall n \geq M$.

If $n \geq \max(N, M)$ then

$$d(x,y) \leq d(x, x_n) + d(x_n, y) < \varepsilon + \varepsilon = \frac{d(x,y)}{2}$$

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QUESTIONS?

COMMENTS?