THE BANACH FIXED POINT THEOREM

Much of mathematics boils down to solving equations – to reformulate your problem into:

find $\bar{x}$ such that $f(\bar{x}) = y$.

- We do this by finding approximations $x_1, x_2, \dots$ so that

  "$f(x_n) \approx y$" or $f(x_n) \xrightarrow[n \to +\infty]{} y$ (consistency)

- It might be possible to prove that $\{x_n\}_n$ is Cauchy.
- If (X, d) is complete then $\{x_n\}_n$ converges to some $\bar{x}$
- If f is continuous then $\bar{x}$ solves the equation:

$$f(\bar{x}) = f\left(\lim_{n \to +\infty} x_n\right) = \lim_{n \to +\infty} f(x_n) = y.$$

- Let (X, d) be a metric space
- Let $f : X \to X$ be a function
- We wish to solve the equation $x = f(x)$

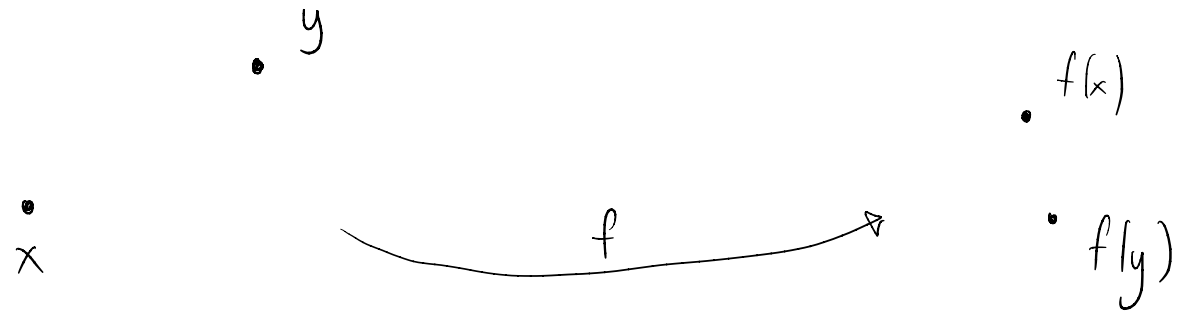
$x \in X$ is a fixed point for f if $x = f(x)$

Strategy: Fixed point iteration (NO: fikspunktiterasjon)

- Let $x_0 \in X$
- Let $x_1 = f(x_0)$
- Let $x_2 = f(x_1)$
- ⋮
- Let $x_{n+1} = f(x_n)$
- ⋮

Let (X,d) be a metric space. A function $f: X \to X$ is a <u>contraction</u> if there is some $k \in (0,1)$ such that

$$d(f(x), f(y)) \leq k\, d(x,y) \qquad \forall\ x,y \in X.$$



<u>Exercise</u>: Every contraction is continuous.

Banach's fixed point theorem

Let (X, d) be a complete metric space and $f: X \to X$ a contraction. Then f has a unique fixed point $\bar{x}$.

Moreover, if $x_0 \in X$ and $x_{n+1} = f(x_n)$ $(n = 0, 1, 2, \ldots)$ then

$$x_n \xrightarrow[n \to \infty]{} \bar{x}.$$

Banach's fixed point theorem

Let (X, d) be a complete metric space and $f: X \to X$ a contraction. Then f has a unique fixed point $\bar{x}$. Moreover, if $x_0 \in X$ and $x_{n+1} = f(x_n)$ $(n = 0, 1, 2, \dots)$ then

$$x_n \xrightarrow[n \to \infty]{} \bar{x}.$$

Proof: • **Claim:** $\{x_n\}_n$ is Cauchy. Indeed,

$$d(x_{n+1}, x_n) = d\big(f(x_n), f(x_{n-1})\big) \le k\, d(x_n, x_{n-1}) \le \dots \le k^n d(x_1, x_0).$$

If, say, $m > n$ then

$$d(x_m, x_n) \le \sum_{\ell=n}^{m-1} d(x_{\ell+1}, x_\ell) \le \sum_{\ell=n}^{m-1} k^\ell\, d(x_1, x_0) = d(x_1, x_0) \sum_{\ell=n}^{m-1} k^\ell$$

$$\le d(x_1, x_0) \sum_{\ell=n}^{\infty} k^\ell = d(x_1, x_0)\, \frac{k^n}{1-k}$$

Thus, if $n, m \geq N$ then

$$d(x_m, x_n) \leq d(x_1, x_0) \frac{k^{\min(n,m)}}{1-k} \leq d(x_1, x_0) \frac{k^N}{1-k}.$$

The latter goes to zero as $N \to \infty$, so $\{x_n\}_n$ is Cauchy.

- Claim: $x_n \xrightarrow[n\to\infty]{} \bar{x}$, a fixed point for f.

Since (X, d) is complete, $\{x_n\}_n$ is convergent, so $x_n \xrightarrow[n\to\infty]{} \bar{x}$ for some $\bar{x} \in X$. Then

$$\bar{x} = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(\bar{x}).$$

- Claim: $\bar{x}$ is the unique fixed point.

  Let $\tilde{x}$ be another fixed point. Then

  $$d(\bar{x}, \tilde{x}) = d\left(f(\bar{x}), f(\tilde{x})\right) \leq k\, d(\bar{x}, \tilde{x}).$$

  Since $k < 1$, we must have $d(\bar{x}, \tilde{x}) = 0$, so $\bar{x} = \tilde{x}$. ∎

QUESTIONS?

COMMENTS?