MODES OF CONTINUITY

Let (X, d_X) and (Y, d_Y) be metric spaces.

A function $f: X \to Y$ is

- <u>continuous at $x \in X$</u> if $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ such that
$$d_Y(f(x), f(y)) < \varepsilon \quad \forall\ y \in X \text{ with } d_X(x, y) < \delta$$
- <u>continuous</u> if it is continuous at all $x \in X$
- <u>uniformly continuous</u> if $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ such that
$$d_Y(f(x), f(y)) < \varepsilon \quad \forall\ x, y \in X \text{ with } d_X(x, y) < \delta$$
- <u>Lipschitz continuous</u> if $\exists\ C > 0$ such that
$$d_Y(f(x), f(y)) \leq C\, d_X(x, y) \quad \forall\ x, y \in X$$

Denote

$C(X,Y) = \{$ all continuous functions $f: X \to Y \}$.

$C(X) = C(X, \mathbb{R})$

Lipschitz continuity $\Rightarrow$ uniform continuity $\Rightarrow$ continuity

Proof: Exercise!

The opposites are not always true.

If $f: A \to \mathbb{R}$ $(A \subseteq \mathbb{R})$ satisfies $|f'(x)| \leq M$ $\forall\, x \in A$, then f is Lipschitz with constant M.

Proof: $|f(x) - f(y)| = |f'(\xi)||x-y| \leq M|x-y|$ $\forall\, x, y \in A$. ∎

If $f: [a,b] \to \mathbb{R}$ is continuously differentiable, then it is Lipschitz.

Proof: $f': [a,b] \to \mathbb{R}$ is continuous and $[a,b]$ is compact, so there is some $M > 0$ so that $|f'(x)| \leq M$ $\forall\, x \in [a,b]$. ∎

Example: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is continuous, but not uniformly continuous.

If $\varepsilon > 0$, $x, y \in \mathbb{R}$, $|x-y| < \delta$ then

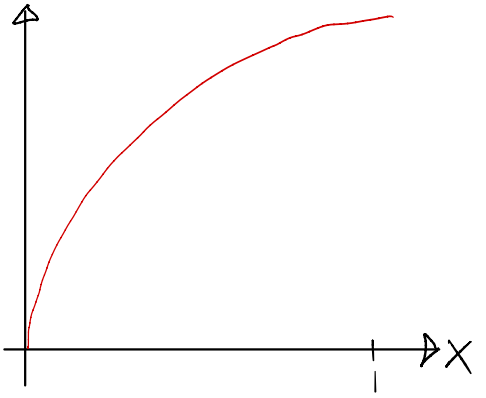
$$|f(x) - f(y)| = |x^2 - y^2| = |x+y| \cdot |x-y| \overset{!}{\leq} \varepsilon$$

If, say, $|x-y| = \frac{\delta}{2}$ then $|f(x) - f(y)| = \frac{\delta}{2} |x+y|$, so if $x, y \geq \frac{\varepsilon}{\delta}$ then

$$|f(x) - f(y)| \geq \frac{\delta}{2} \cdot \frac{2\varepsilon}{\delta} = \varepsilon \not< \varepsilon.$$

Example: $f: [0,1] \to \mathbb{R}$, $f(x) = \sqrt{x}$ is uniformly continuous, but not Lipschitz

- f is differentiable, but $f'(x) = \frac{1}{2\sqrt{x}}$ is not bounded !



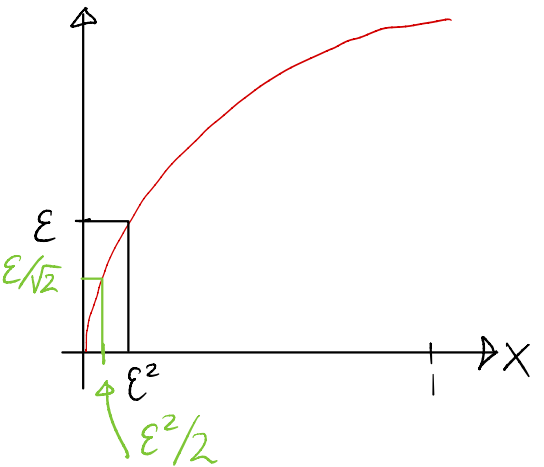
Example: $f:[0,1]\to\mathbb{R}$, $f(x)=\sqrt{x}$ is uniformly continuous, but not Lipschitz

- f is differentiable, but $f'(x)=\frac{1}{2\sqrt{x}}$ is not bounded!

Let $\delta=\varepsilon^2/2$, let $x,y\in[0,1]$, $|x-y|<\delta$.

(a) If $x,y<\varepsilon^2$ then $0\le f(x),f(y)<\varepsilon$, so $|f(x)-f(y)|<\varepsilon$

(b) If, say, $x\ge\varepsilon^2$ then $y>\varepsilon^2-\delta=\varepsilon^2/2$. We get, for some ξ,



$$|f(x)-f(y)|=\frac{1}{2\sqrt{\xi}}|x-y|<\frac{1}{2\sqrt{\min(x,y)}}\delta<\frac{1}{2\sqrt{\varepsilon^2/2}}\cdot\frac{\varepsilon^2}{2}=\frac{\varepsilon}{2\sqrt{2}}<\varepsilon.$$

$\Rightarrow$ f is unif. continuous

Example: $f: [0,1] \to \mathbb{R}$, $f(x) = \sqrt{x}$ is uniformly continuous, but not Lipschitz

- f is not Lipschitz: if $x = 0$ and $y > 0$ then

$$\frac{|f(x) - f(y)|}{|x - y|} = \frac{|\sqrt{0} - \sqrt{y}|}{|0 - y|} = \frac{1}{\sqrt{y}}$$

which cannot be bounded by a single constant $C > 0$.

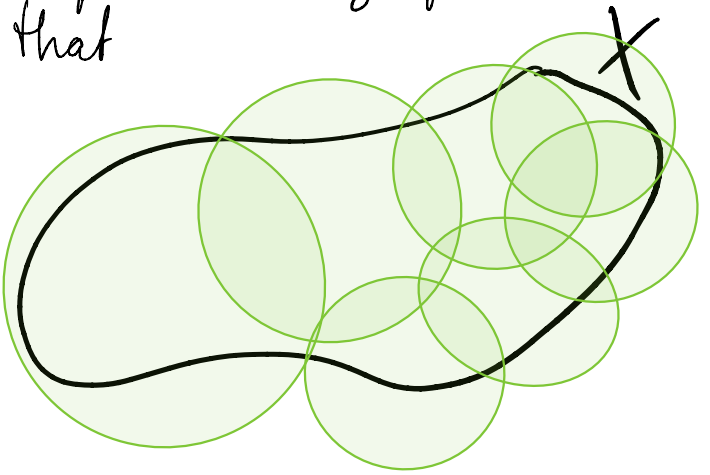
Let $f: X \to Y$ be continuous and (X, d_X) compact. Then f is uniformly continuous.

Proof: Let $\varepsilon > 0$, and for each $x \in X$, let $\delta_x > 0$ be s.t.

$$d_Y(f(x), f(y)) < \varepsilon \quad \text{when } d_X(x, y) < \delta_x.$$

Then $\{B(x; \delta_x/2) : x \in X\}$ is an open covering of X, so there are $x_1, \ldots, x_n \in X$ such that

$$X = \bigcup_{i=1}^{n} B(x_i; \delta_{x_i}/2)$$



Let $f: X \to Y$ be continuous and (X, d_X) compact. Then f is uniformly continuous.

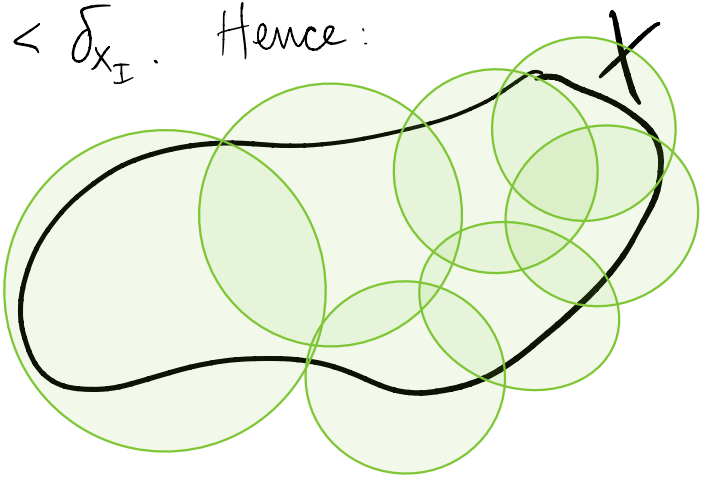
Proof (cont.):

$$X = \bigcup_{i=1}^{n} B(x_i; \delta_{x_i}/2)$$

Let $\delta = \min\{\delta_{x_i}/2 : i = 1, \dots, n\}$. Let $x, y \in X$ with $d_X(x,y) < \delta$. There is some $I \in \{1, \dots, n\}$ s.t. $x \in B(x_I; \delta_{x_I}/2)$.

Then $d_X(x_I, y) \leq d_X(x_I, x) + d_X(x,y) < \delta_{x_I}$. Hence:

$$d_Y(f(x), f(y)) \leq d_Y(f(x), f(x_I)) + d_Y(f(x_I), f(y))$$
$$< \varepsilon + \varepsilon \leq 2\varepsilon.$$



∎

QUESTIONS?

COMMENTS?