WEIERSTRASS' APPROXIMATION THEOREM

I

Let $f: X \to Y$ be any function.

The approximation problem

Can we find "nicer" functions $f_1, f_2, \ldots : X \to Y$ such that $f_n \xrightarrow[n \to \infty]{} f$?

Application: Numerical approximation

To approximate the integral/derivative/... of $f: \mathbb{R} \to \mathbb{R}$, replace f by a polynomial $f_n \approx f$ and compute with f_n instead.

Let $f: X \to Y$ be any function.

The approximation problem

Can we find "nicer" functions $f_1, f_2, \ldots : X \to Y$ such that $f_n \xrightarrow[n \to \infty]{} f$?

- For concreteness we let $X = [a, b]$, $Y = \mathbb{R}$.
- We need to assume some regularity on f. Assume f continuous.
- We will find polynomials $f_1, f_2, \ldots$

Polynomial approximation

If $f:[a,b]\to\mathbb{R}$ is smooth and $x_0\in(a,b)$ then

$$f=f_n+R_n,\qquad f_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k,\qquad R_n(x)\sim\frac{\|f^{(n+1)}\|_\infty}{(n+1)!}|x-x_0|^{n+1}$$

so if f is smooth enough (real analytic) then

$$f_n\xrightarrow[n\to\infty]{}f \quad \text{uniformly}$$

But what if f is merely continuous?

We first transform $[a,b]$ to $[0,1]$:

$$y \in [a,b] \longmapsto x = \frac{y-a}{b-a} \in [0,1]$$

$$\Longleftrightarrow$$

$$x \in [0,1] \longmapsto y = a + (b-a)x.$$

Thus, $f \in C([a,b])$ is transformed to

$$g(x) = f\left(a + (b-a)x\right)$$

$$\Longleftrightarrow \quad f(y) = g\left(\frac{y-a}{b-a}\right).$$

Then $g \in C([0,1])$.

Def: Let $n \in \mathbb{N}_0$. The space of n-th order polynomials is

$$\mathbb{P}_n = \{ p : \mathbb{R} \to \mathbb{R} \; : \; p(x) = a_0 + a_1 x + \cdots + a_n x^n \quad \forall \; x \in \mathbb{R} \text{ for } a_0, \ldots, a_n \in \mathbb{R} \}.$$

We write $\mathbb{P} = \bigcup_{n \in \mathbb{N}_0} \mathbb{P}_n$.

Note that:

- $\mathbb{P}_0 \subseteq \mathbb{P}_1 \subseteq \mathbb{P}_2 \subseteq \ldots$
- $\mathbb{P}_n \subseteq \mathbb{P} \quad \forall \; n \in \mathbb{N}_0$
- $\mathbb{P}$ and $\mathbb{P}_n$ are vector spaces for all $n \in \mathbb{N}_0$
- $\dim \mathbb{P}_n = n+1$.

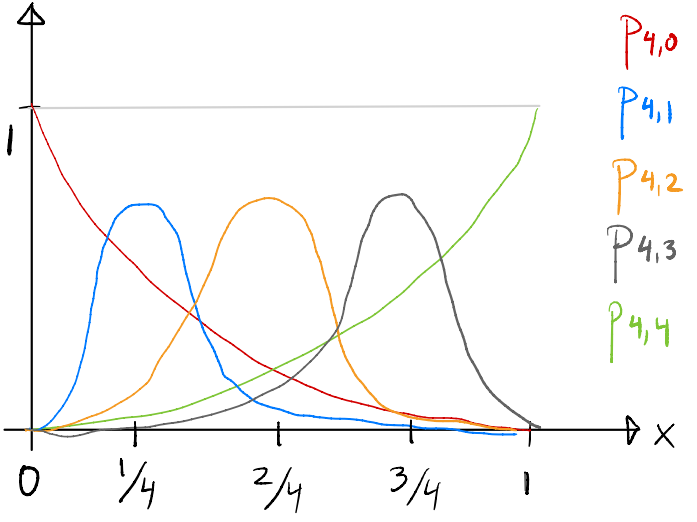
Def: The **Bernstein polynomials** are

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad \text{for } x \in [0,1]$$

for $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n$.

Recall: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Note: $p_{n,k} \in \mathbb{P}_n \quad \forall\, n, k.$



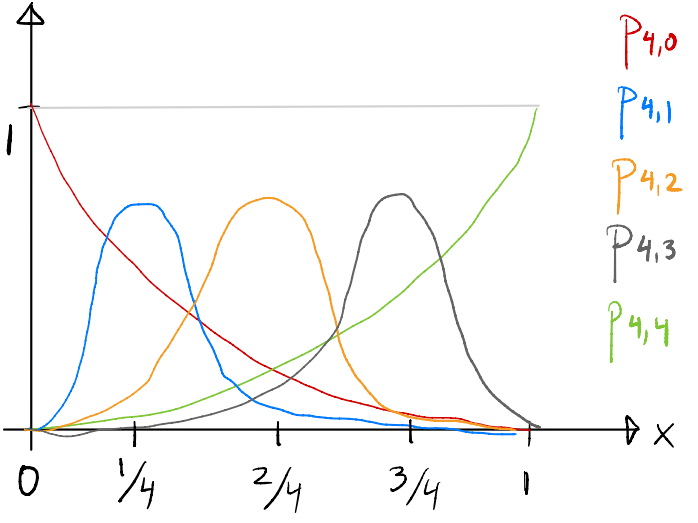
Def: The **Bernstein polynomials** are

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad \text{for } x \in [0,1]$$

for $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n$.

Fact 1:

$$p_{n,k}(x) \geq 0 \quad \forall \; x \in [0,1]$$



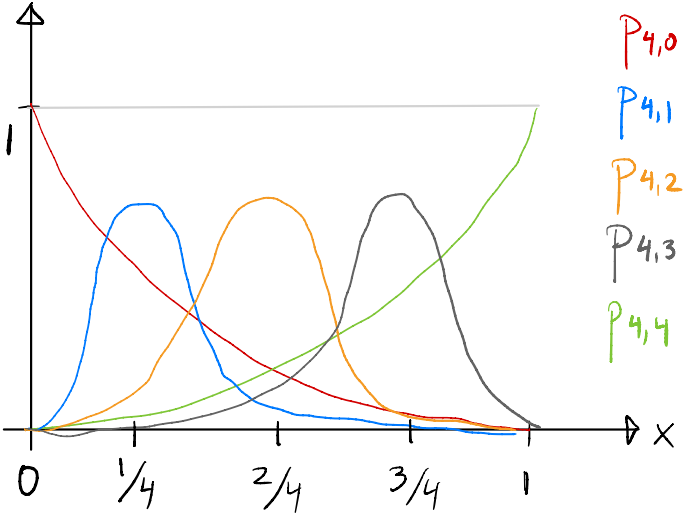
Def: The Bernstein polynomials are

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad \text{for } x \in [0,1]$$

for $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n$.

Fact 2:

$p_{n,k}$ has a single maximum at $x_{n,k} = k/n$



Def: The **Bernstein polynomials** are

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \qquad \text{for } x \in [0,1]$$

for $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n$.

Fact 3: $\sum_{k=0}^{n} p_{n,k}(x) = 1 \quad \forall \; x \in [0,1]$. In particular, $p_{n,k}(x) \leq 1$.

Use the binomial theorem: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Set $a = x$, $b = 1-x$:

$$\sum_{k=0}^{n} p_{n,k}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = \left(x + (1-x)\right)^n = 1^n = 1.$$

Def: The **Bernstein polynomials** are

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \qquad \text{for } x \in [0,1]$$

for $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n$.

Fact 3: $\sum_{k=0}^{n} p_{n,k}(x) = 1 \quad \forall \; x \in [0,1]$. In particular, $p_{n,k}(x) \leq 1$.

(The Bernstein polynomials $p_{n,0}, \ldots, p_{n,n}$ form a partition of unity.)

Def: The **Bernstein polynomials** are

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \qquad \text{for } x \in [0,1]$$

for $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n$.

Fact 4: $\sum_{k=0}^{n} p_{n,k}(x)\, k = nx \quad \forall \; x \in [0,1]$.

Differentiate $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ with respect to a:

$$n(a+b)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k\, a^{k-1} b^{n-k} = \frac{1}{a} \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \cdot k.$$

Now set $a = x$, $b = 1 - x$.

Def: The Bernstein polynomials are

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \qquad \text{for } x \in [0,1]$$

for $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n$.

Fact 5: $\sum_{k=0}^{n} p_{n,k}(x) k^2 = xn + x^2 n(n-1) \qquad \forall\ x \in [0,1]$

Differentiate $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ twice with respect to a:

$$n(n-1)(a+b)^{n-2} = \frac{1}{a^2} \sum_{k=0}^{n} \binom{n}{k} k(k-1) a^k b^{n-k}.$$

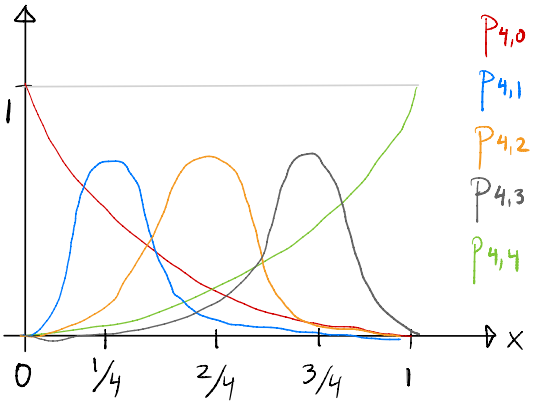
Now set $a = x$, $b = 1-x$ and use Fact 4.

Def: The **Bernstein polynomials** are

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad \text{for } x \in [0,1]$$

for $n \in \mathbb{N}$ and $k = 0, 1, \dots, n$.

- $p_{n,k}(x) \in [0,1] \quad \forall \; x \in [0,1]$
- $p_{n,k}$ has a single max. at $x_{n,k} = k/n$
- $\sum_{k=0}^{n} p_{n,k}(x) = 1 \quad \forall \; x \in [0,1]$
- $\sum_{k=0}^{n} p_{n,k}(x)\, k = nx \quad \forall \; x \in [0,1]$
- $\sum_{k=0}^{n} p_{n,k}(x)\, k^2 = xn + x^2 n(n-1) \quad \forall \; x \in [0,1]$



Def:

Let $f : [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$. The n-th Bernstein approximation of f is $f_n \in \mathbb{P}_n$ given by

$$f_n(x) = \sum_{k=0}^{n} p_{n,k}(x)\, f(x_{n,k})$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

Aim: Prove that $f_n \xrightarrow[n \to \infty]{} f$ uniformly.

QUESTIONS?

COMMENTS?

Next video: Prove that $f_n \xrightarrow[n \to \infty]{} f$.